IDR versus other Krylov subspace solvers

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joint work with Olaf Rendel & Anisa Rizvanolli

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Outline

Krylov subspace methods

Hessenberg decompositions

Polynomial representations

IDR

IDR, IDR(s), and IDREIG

IDR vs. other Krylov subspace methods

IDRSTAB and QMRIDR

Transferring techniques

Stay close to Arnoldi/Lanczos

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Introduction

Krylov subspace methods: approximations

$$\left.\begin{matrix} \mathbf{x}_k, \underline{\mathbf{x}}_k, \\ \mathbf{y}_k, \underline{\mathbf{y}}_k \end{matrix}\right\} \in \mathcal{K}_k(\mathbf{A}, \mathbf{q}) := \text{span}\,\{\mathbf{q}, \mathbf{A}\mathbf{q}, \dots, \mathbf{A}^{k-1}\mathbf{q}\} = \{p(\mathbf{A})\mathbf{q} \mid p \in \mathbb{P}_{k-1}\},$$

where

$$\mathbb{P}_{k-1} := \Big\{\sum_{j=0}^{k-1} \alpha_j z^j \mid \alpha_j \in \mathbb{C},\ 0 \leqslant j < k\Big\},$$

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to solutions of linear systems

$$\mathbf{A}\mathbf{x} = \mathbf{r}_0 \ (= \mathbf{b} - \mathbf{A}\mathbf{x}_0), \qquad \mathbf{A} \in \mathbb{C}^{n\times n}, \quad \mathbf{b}, \mathbf{x}_0 \in \mathbb{C}^n,$$

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and (partial) eigenproblems

$$\mathbf{A}\mathbf{v} = \mathbf{v}\lambda, \qquad \mathbf{A} \in \mathbb{C}^{n\times n}.$$

Hessenberg decompositions

Construction of basis vectors resembled in structure of arising Hessenberg decomposition

$$\mathbf{A}\mathbf{Q}_k = \mathbf{Q}_{k+1}\underline{\mathbf{H}}_k,$$

where

- $\mathbf{Q}_{k+1} = \left(\mathbf{Q}_k, \mathbf{q}_{k+1}\right) \in \mathbb{C}^{n\times(k+1)}$ collects basis vectors,
- $\underline{\mathbf{H}}_k \in \mathbb{C}^{(k+1)\times k}$ is unreduced extended Hessenberg.

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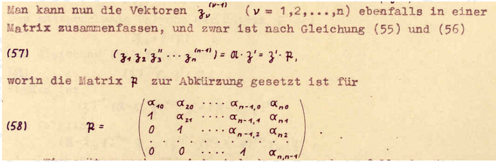
Aspects of perturbed Krylov subspace methods: captured with perturbed Hessenberg decompositions

$$\mathbf{A}\mathbf{Q}_k + \mathbf{F}_k = \mathbf{Q}_{k+1}\underline{\mathbf{H}}_k,$$

$\mathbf{F}_k \in \mathbb{C}^{n\times k}$ accounts for perturbations (finite precision & inexact methods).

Karl Hessenberg & “his” matrix + decomposition

"‘Behandlung linearer Eigenwertaufgaben mit Hilfe der Hamilton-Cayleyschen Gleichung"’, Karl Hessenberg, 1. Bericht der Reihe "‘Numerische Verfahren"’, July, 23rd 1940, page 23:



Man kann nun die Vektoren $\mathfrak{z}_\nu^{(\nu-1)}$ $(\nu = 1,2,\dots,n)$ ebenfalls in einer Matrix zusammenfassen, und zwar ist nach Gleichung (55) und (56)

(57) $$(\mathfrak{z}_1 \mathfrak{z}_2' \mathfrak{z}_3'' \cdots \mathfrak{z}_n^{(n-1)}) = \mathfrak{A}\cdot\mathfrak{z}' = \mathfrak{z}'\cdot\mathfrak{P},$$

worin die Matrix $\mathfrak{P}$ zur Abkürzung gesetzt ist für

(58) $$\mathfrak{P} = \begin{pmatrix} \alpha_{10} & \alpha_{20} & \cdots & \alpha_{n-1,0} & \alpha_{n0} \\ 1 & \alpha_{21} & \cdots & \alpha_{n-1,1} & \alpha_{n1} \\ 0 & 1 & \cdots & \alpha_{n-1,2} & \alpha_{n2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{n,n-1} \end{pmatrix}.$$

- Hessenberg decomposition, Eqn. (57),
- Hessenberg matrix, Eqn. (58).

Karl Hessenberg (* September 8th, 1904, † February 22nd, 1959)

Important Polynomials

Residuals of OR and MR approximation

$$\mathbf{x}_k := \mathbf{Q}_k \mathbf{z}_k \qquad \text{and} \qquad \underline{\mathbf{x}}_k := \mathbf{Q}_k \underline{\mathbf{z}}_k$$

with coefficient vectors

$$\mathbf{z}_k := \mathbf{H}_k^{-1} \mathbf{e}_1 \|\mathbf{r}_0\| \qquad \text{and} \qquad \underline{\mathbf{z}}_k := \underline{\mathbf{H}}_k^{\dagger} \underline{\mathbf{e}}_1 \|\mathbf{r}_0\|$$

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$$\mathcal{R}_k(z) := \det\left(\mathbf{I}_k - z\mathbf{H}_k^{-1}\right) \quad \text{and} \quad \underline{\mathcal{R}}_k(z) := \det\left(\mathbf{I}_k - z\underline{\mathbf{H}}_k^{\dagger}\underline{\mathbf{I}}_k\right).$$

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Convergence of OR and MR depends on (harmonic) Ritz values.

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IDR(s) is a Krylov subspace method $\rightsquigarrow$ all techniques from 90's applicable!

IDR(s)

IDR spaces:

$$\mathcal{G}_0 := \mathcal{K}(\mathbf{A}, \mathbf{q}), \qquad \text{(full Krylov subspace)}$$
$$\mathcal{G}_j := (\alpha_j \mathbf{A} + \beta_j \mathbf{I})(\mathcal{G}_{j-1} \cap \mathcal{S}), \quad j \geqslant 1, \quad \alpha_j, \beta_j \in \mathbb{C}, \quad \alpha_j \neq 0,$$

where

$$\text{codim}(\mathcal{S}) = s, \quad \text{e.g.,} \quad \mathcal{S} = \text{span}\,\{\widetilde{\mathbf{R}}_0\}^{\perp}, \quad \widetilde{\mathbf{R}}_0 \in \mathbb{C}^{n \times s}.$$

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Interpreted as Sonneveld spaces (Sleijpen, Sonneveld, van Gijzen 2010):

$$\mathcal{G}_j = \mathcal{S}_j(P_j, \mathbf{A}, \widetilde{\mathbf{R}}_0) := \left\{ P_j(\mathbf{A})\mathbf{v} \mid \mathbf{v} \perp \mathcal{K}_j(\mathbf{A}^\mathsf{H}, \widetilde{\mathbf{R}}_0) \right\},$$
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Image of shrinking space: Induced Dimension Reduction.

IDR(s)

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$$\mathbf{v}_k = \sum_{i=k-s}^{k} \mathbf{g}_i \gamma_i, \quad \widetilde{\mathbf{R}}_0^{\mathsf{H}} \mathbf{v}_k = \mathbf{o}_s, \quad k \geqslant s+1,$$

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$$\mathbf{g}_{k+1}\nu_{k+1} = (\alpha_j \mathbf{A} + \beta_j \mathbf{I})\mathbf{v}_k - \sum_{i=k-j(s+1)-1}^{k} \mathbf{g}_i \nu_i, \quad j = \left\lfloor \frac{k-1}{s+1} \right\rfloor.$$

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Generalized Hessenberg decomposition:

$$\mathbf{A}\mathbf{V}_k = \mathbf{A}\mathbf{G}_k\mathbf{U}_k = \mathbf{G}_{k+1}\underline{\mathbf{H}}_k,$$

where $\mathbf{U}_k \in \mathbb{C}^{k\times k}$ upper triangular.

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Structure of Sonneveld pencils:

$$\mathbf{H}_k = \begin{pmatrix}
\times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
+ & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & + & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & + & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ \\
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\circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times & \times & \times & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times & \times & \times \\
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\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times
\end{pmatrix}, \quad
\begin{pmatrix}
\times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \times & \times & \times & \times & \circ & \circ & \circ & \circ \\
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\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times
\end{pmatrix} = \mathbf{U}_k$$

IDREIG

Eigenvalues of Sonneveld pencil $(\mathbf{H}_k, \mathbf{U}_k)$ are roots of residual polynomials. Those distinct from roots of

$$P_j(z) = \prod_{i=1}^{j} (\alpha_i z + \beta_i), \quad \text{i.e.,} \quad z_i = -\frac{\beta_i}{\alpha_i}, \quad 1 \leqslant i \leqslant j$$

converge to eigenvalues of $\mathbf{A}$.

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Suppose $\mathbf{G}_{k+1}$ of full rank. Sonneveld pencil $(\mathbf{H}_k, \mathbf{U}_k)$ as oblique projection:

$$\begin{aligned} \widehat{\mathbf{G}}_k^{\mathsf{H}}(\mathbf{A}, \mathbf{I}_n)\mathbf{G}_k\mathbf{U}_k &= \widehat{\mathbf{G}}_k^{\mathsf{H}}(\mathbf{A}\mathbf{G}_k\mathbf{U}_k, \mathbf{G}_k\mathbf{U}_k) \\ &= \widehat{\mathbf{G}}_k^{\mathsf{H}}(\mathbf{G}_{k+1}\underline{\mathbf{H}}_k, \mathbf{G}_k\mathbf{U}_k) = (\underline{\mathbf{I}}_k^{\mathsf{T}}\underline{\mathbf{H}}_k, \mathbf{U}_k) = (\mathbf{H}_k, \mathbf{U}_k), \end{aligned} \qquad (1)$$

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Use deflated pencil for Lanczos Ritz values (Gutknecht, Z. (2010): IDREIG).

IDREIG

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Use deflated pencil for Lanczos Ritz values (Gutknecht, Z. (2010): IDREIG). First: IDR(s)ORES, Olaf Rendel: IDR(s)BIO, Anisa Rizvanolli: IDR(s)STAB(ℓ).

Outline

Krylov subspace methods

- Hessenberg decompositions
- Polynomial representations

IDR

- IDR, IDR(s), and IDREIG

IDR vs. other Krylov subspace methods

- IDRSTAB and QMRIDR
- Transferring techniques
- Stay close to Arnoldi/Lanczos

IDRSTAB

IDR(s)STAB(ℓ) (Tanio & Sugihara; Sleijpen & van Gijzen): combine ideas of IDR(s) and BiCGSTAB(ℓ).

IDRSTAB

IDR(s)STAB(ℓ) (Tanio & Sugihara; Sleijpen & van Gijzen): combine ideas of IDR(s) and BICGSTAB(ℓ).

IDRSTAB (Sleijpen's implementation) recursively computes "(extended) Hessenberg matrices of basis matrices and residuals" ($k \geqslant 1$):

$$
\begin{array}{llll}
\mathbf{G}_{11}^{(k)}, \mathbf{r}_{11}^{(k)} & \mathbf{G}_{12}^{(k)}, \mathbf{r}_{12}^{(k)} & \dots & \mathbf{G}_{1,\ell+1}^{(k)}, \quad \mathbf{r}_{1,\ell+1}^{(k)} \\
\mathbf{G}_{21}^{(k)}, \mathbf{r}_{21}^{(k)} & \mathbf{G}_{22}^{(k)}, \mathbf{r}_{22}^{(k)} & \dots & \mathbf{G}_{2,\ell+1}^{(k)}, \quad \mathbf{r}_{2,\ell+1}^{(k)} \\
 & \mathbf{G}_{32}^{(k)}, \mathbf{r}_{32}^{(k)} & \ddots & \qquad \vdots \\
 & & \ddots & \mathbf{G}_{\ell+1,\ell+1}^{(k)}, \mathbf{r}_{\ell+1,\ell+1}^{(k)} \\
 & & & \mathbf{G}_{\ell+2,\ell+1}^{(k)}
\end{array}
$$

$$
\begin{aligned}
\mathbf{G}_{i,j}^{(k)} &\in \mathbb{C}^{n \times s}, & \mathbf{r}_{i,j}^{(k)} &\in \mathbb{C}^{n}, \\
\mathbf{G}_{i+1,j}^{(k)} &= \mathbf{A}\mathbf{G}_{i,j}^{(k)}, & \mathbf{r}_{i+1,j}^{(k)} &= \mathbf{A}\mathbf{r}_{i,j}^{(k)}, \\
\widetilde{\mathbf{R}}_0^{\mathsf{H}}\mathbf{G}_{ii}^{(k)} &= \mathbf{O}_s, & \widetilde{\mathbf{R}}_0^{\mathsf{H}}\mathbf{r}_{ii}^{(k)} &= \mathbf{o}_s, \\
(\mathbf{G}_{ii}^{(k)})^{\mathsf{H}}\mathbf{G}_{ii}^{(k)} &= \mathbf{I}_s.
\end{aligned}
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$$

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(\mathbf{G}_{ii}^{(k)})^{\mathsf{H}}\mathbf{G}_{ii}^{(k)} &= \mathbf{I}_s.
\end{aligned}
$$

Initialization using Arnoldi's method:

$$
\mathbf{G}_{21}^{(1)} = \mathbf{A}\mathbf{G}_{11}^{(1)} = (\mathbf{G}_{11}^{(1)}, \mathbf{g}_{\mathrm{tmp}})\underline{\mathbf{H}}_s^{(0)},
$$

$$
\mathbf{r}_{11}^{(1)} = \mathbf{r}_0 - \mathbf{G}_{21}^{(1)}\boldsymbol{\alpha}^{(1)} = (\mathbf{I} - \mathbf{G}_{21}^{(1)}(\widetilde{\mathbf{R}}_0^{\mathsf{H}}\mathbf{G}_{21}^{(1)})^{-1}\widetilde{\mathbf{R}}_0^{\mathsf{H}})\mathbf{r}_0, \quad \mathbf{r}_{21}^{(1)} = \mathbf{A}\mathbf{r}_{11}^{(1)}.
$$

IDRSTAB

Columnwise update (IDR part) such that diagonal blocks

- form basis of $\mathcal{G}_j \setminus \mathcal{G}_{j+1}$ with expansion $\mathcal{G}_j = \mathbf{A}(\mathcal{G}_{j-1} \cap \mathcal{S}) \rightsquigarrow \beta^{(j)} \in \mathbb{C}^{s \times s}$,
- are orthonormalized $\rightsquigarrow \underline{\mathbf{H}}_{s-1}^{(j)} \in \mathbb{C}^{s \times (s-1)}$

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In particular, with $\widetilde{\mathbf{v}}_i \in \mathcal{G}_{j-1} \cap \mathcal{S}$,

$$\begin{aligned} \beta_i^{(j)} &= (\widetilde{\mathbf{R}}_0^{\mathsf{H}} \mathbf{G}_{j,j-1})^{-1} \widetilde{\mathbf{R}}_0^{\mathsf{H}} (\mathbf{A}\widetilde{\mathbf{v}}_i) \\ \Rightarrow \quad (\mathbf{A}\widetilde{\mathbf{v}}_i) - \mathbf{G}_{j,j-1}\beta_i^{(j)} &= \mathbf{A}(\widetilde{\mathbf{v}}_i - \mathbf{G}_{j-1,j-1}\beta_i^{(j)}) \in \mathcal{G}_j \cap \mathcal{S} \end{aligned}$$

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Every new vector in $\mathcal{G}_j \cap \mathcal{S}$ is orthonormalized with respect to the others.

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$$\Rightarrow \quad (\mathbf{A}\widetilde{\mathbf{v}}_i) - \mathbf{G}_{j,j-1}\beta_i^{(j)} = \mathbf{A}(\widetilde{\mathbf{v}}_i - \mathbf{G}_{j-1,j-1}\beta_i^{(j)}) \in \mathcal{G}_j \cap \mathcal{S}$$

Every new vector in $\mathcal{G}_j \cap \mathcal{S}$ is orthonormalized with respect to the others.

Thus, for the IDR-IDRSTAB pencil relating (STAB-purified) diagonal blocks,

- $\beta^{(j)} \in \mathbb{C}^{s \times s}$ couples $\mathbf{G}_{jj}$ and $\mathbf{G}_{j,j-1} = \mathbf{A}\mathbf{G}_{j-1,j-1} \rightsquigarrow \mathbf{U}_k$,
- $\underline{\mathbf{H}}_{s-1}^{(j)} \in \mathbb{C}^{s \times (s-1)}$ couples result with others in same block $\rightsquigarrow \underline{\mathbf{H}}_k$.

IDRSTAB

Columnwise update (IDR part) such that diagonal blocks

- form basis of $\mathcal{G}_j \setminus \mathcal{G}_{j+1}$ with expansion $\mathcal{G}_j = \mathbf{A}(\mathcal{G}_{j-1} \cap \mathcal{S}) \rightsquigarrow \beta^{(j)} \in \mathbb{C}^{s \times s}$,
- are orthonormalized $\rightsquigarrow \underline{\mathbf{H}}_{s-1}^{(j)} \in \mathbb{C}^{s \times (s-1)}$

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All other blocks in column treated in same manner.

IDRSTAB

Residual updates en détail ($i \leqslant j$, $\mathbf{r}_{j+1,j}^{(k)} = \mathbf{A}\mathbf{r}_{j,j}^{(k)}$):

$$\mathbf{r}_{i,j}^{(k)} = \mathbf{r}_{i,j-1}^{(k)} - \mathbf{G}_{i+1,j}^{(k)}\alpha^{(j)}, \quad \mathbf{r}_{j,j}^{(k)} = (\mathbf{I} - \mathbf{G}_{j+1,j}^{(k)}(\widetilde{\mathbf{R}}_0^{\mathsf{H}}\mathbf{G}_{j+1,j}^{(k)})^{-1}\widetilde{\mathbf{R}}_0^{\mathsf{H}})\mathbf{r}_{j,j-1}^{(k)}.$$

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Here,

$$\boldsymbol{\alpha}^{(j)} := (\widetilde{\mathbf{R}}_0^{\mathsf{H}}\mathbf{G}_{j+1,j}^{(k)})^{-1}\widetilde{\mathbf{R}}_0^{\mathsf{H}}\mathbf{r}_{j,j-1}^{(k)},$$

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$\boldsymbol{\alpha}^{(j)}$ relating $\mathbf{r}_{j,j-1}^{(k)} = \mathbf{A}\mathbf{r}_{j-1,j-1}^{(k)}$ (old) and $\mathbf{r}_{j,j}^{(k)}$ (new) via $\mathbf{G}_{j+1,j}^{(k)} = \mathbf{A}\mathbf{G}_{j,j}^{(k)} \rightsquigarrow \mathbf{U}_k$.

IDRStab

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New cycle (Stab part, $\mathbf{r}_{21}^{(k+1)} = \mathbf{A}\mathbf{r}_{11}^{(k+1)}$, $\gamma^{(\ell)} \in \mathbb{C}^s$ such that $\|\mathbf{r}_{11}^{(k+1)}\| = \min$):

$$\mathbf{r}_{11}^{(k+1)} = \mathbf{r}_{1,\ell+1}^{(k)} - \sum_{i=1}^{\ell} \mathbf{r}_{i+1,\ell+1}^{(k)}\gamma_i^{(\ell)}, \qquad \begin{cases} \mathbf{G}_{11}^{(k+1)} = \mathbf{G}_{1,\ell+1}^{(k)} - \sum_{i=1}^{\ell} \mathbf{G}_{i+1,\ell+1}^{(k)}\gamma_i^{(\ell)}, \\ \mathbf{G}_{21}^{(k+1)} = \mathbf{G}_{2,\ell+1}^{(k)} - \sum_{i=1}^{\ell} \mathbf{G}_{i+2,\ell+1}^{(k)}\gamma_i^{(\ell)}. \end{cases}$$

IDRSTAB

Residual updates en détail ($i \leqslant j$, $\mathbf{r}^{(k)}_{j+1,j} = \mathbf{A}\mathbf{r}^{(k)}_{j,j}$):

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Here,

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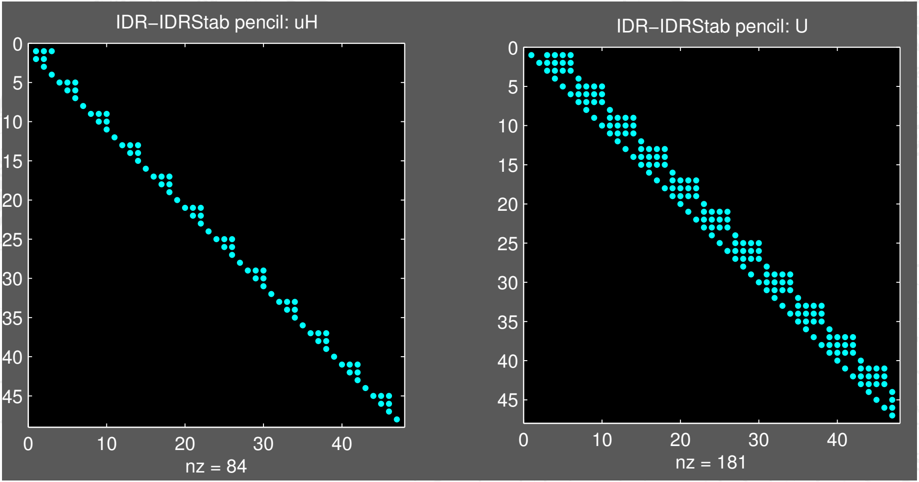
New cycle (STAB part, $\mathbf{r}^{(k+1)}_{21} = \mathbf{A}\mathbf{r}^{(k+1)}_{11}$, $\gamma^{(\ell)} \in \mathbb{C}^s$ such that $\|\mathbf{r}^{(k+1)}_{11}\| = \min$):

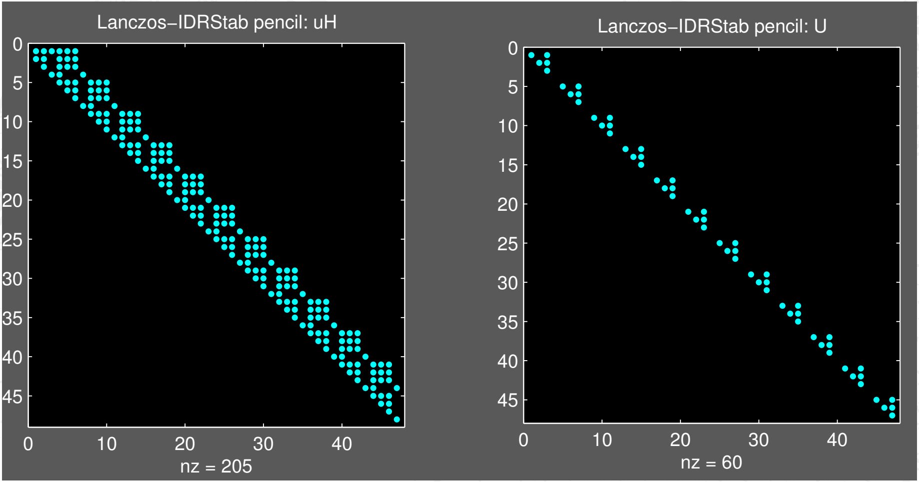
$$\mathbf{r}^{(k+1)}_{11} = \mathbf{r}^{(k)}_{1,\ell+1} - \sum_{i=1}^{\ell} \mathbf{r}^{(k)}_{i+1,\ell+1}\gamma_i^{(\ell)}, \quad \begin{cases} \mathbf{G}^{(k+1)}_{11} = \mathbf{G}^{(k)}_{1,\ell+1} - \sum_{i=1}^{\ell} \mathbf{G}^{(k)}_{i+1,\ell+1}\gamma_i^{(\ell)}, \\ \mathbf{G}^{(k+1)}_{21} = \mathbf{G}^{(k)}_{2,\ell+1} - \sum_{i=1}^{\ell} \mathbf{G}^{(k)}_{i+2,\ell+1}\gamma_i^{(\ell)}. \end{cases}$$

Anisa Rizvanolli: $\rightsquigarrow$ Lanczos-IDRSTAB pencil for eigenvalues, IDRSTABEIG.

Structure of (STAB-purified) IDR-IDRSTAB pencil



Structure of (undeflated) Lanczos-IDRSTAB pencil



QMRIDR

MR methods: use extended Hessenberg matrix

$$\mathbf{x}_k := \mathbf{Q}_k \underline{\mathbf{z}}_k, \quad \underline{\mathbf{z}}_k := \underline{\mathbf{H}}_k^{\dagger} \underline{\mathbf{e}}_1 \|\mathbf{r}_0\|.$$

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IDR based: generalized Hessenberg decomposition,

$$\mathbf{A}\mathbf{V}_k = \mathbf{A}\mathbf{G}_k\mathbf{U}_k = \mathbf{G}_{k+1}\underline{\mathbf{H}}_k.$$

Thus,

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Simplified residual bound (block-wise orthonormalization):

$$\begin{aligned}\|\underline{\mathbf{r}}_k\| = \|\mathbf{r}_0 - \mathbf{A}\underline{\mathbf{x}}_k\| &\leqslant \|\mathbf{G}_{k+1}\| \cdot \left\|\underline{\mathbf{e}}_1\|\mathbf{r}_0\| - \underline{\mathbf{H}}_k\underline{\mathbf{z}}_k\right\| \\ &\leqslant \sqrt{\left\lceil \frac{k+1}{s+1} \right\rceil} \cdot \left\|\underline{\mathbf{e}}_1\|\mathbf{r}_0\| - \underline{\mathbf{H}}_k\underline{\mathbf{z}}_k\right\|.\end{aligned}$$

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Implementation based on short recurrences possible.

QMRIDR

Other Krylov-paradigms possible, e.g., flexible QMRIDR:

QMRIDR

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$$P_j(\mathbf{A})\mathbf{v}_k = (\alpha_j \mathbf{A} + \beta_j \mathbf{I})\mathbf{v}_k \rightsquigarrow (\alpha_j \mathbf{A}\mathbf{P}_k^{-1} + \beta_j \mathbf{I})\mathbf{v}_k = \mathbf{A}\widetilde{\mathbf{v}}_k + \beta_j \mathbf{v}_k,$$

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Computation of flexible MR iterate and flexible MR approximation:

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Flexible IDR variants algorithmically very easy to implement.

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Various extensions for IDRSTAB: Olaf Rendel, Z. $\rightsquigarrow$ QMRIDRSTAB.

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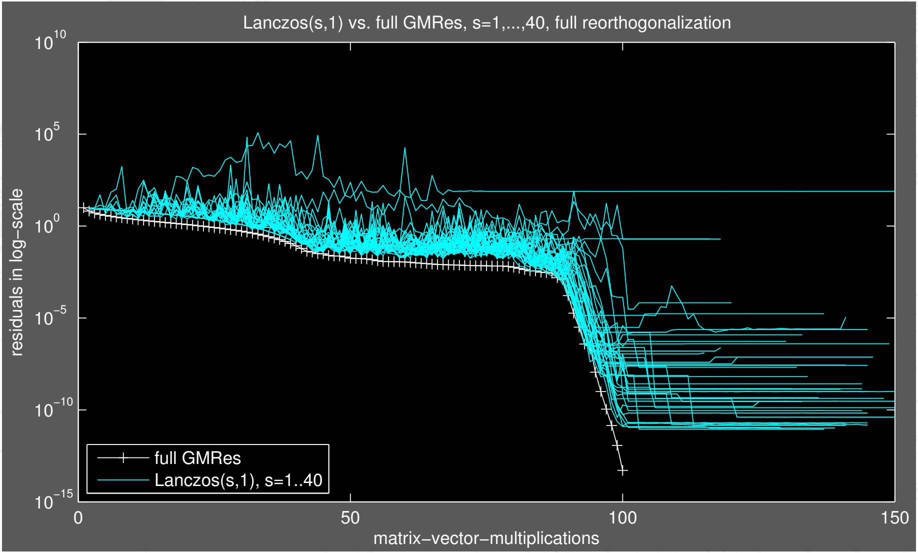
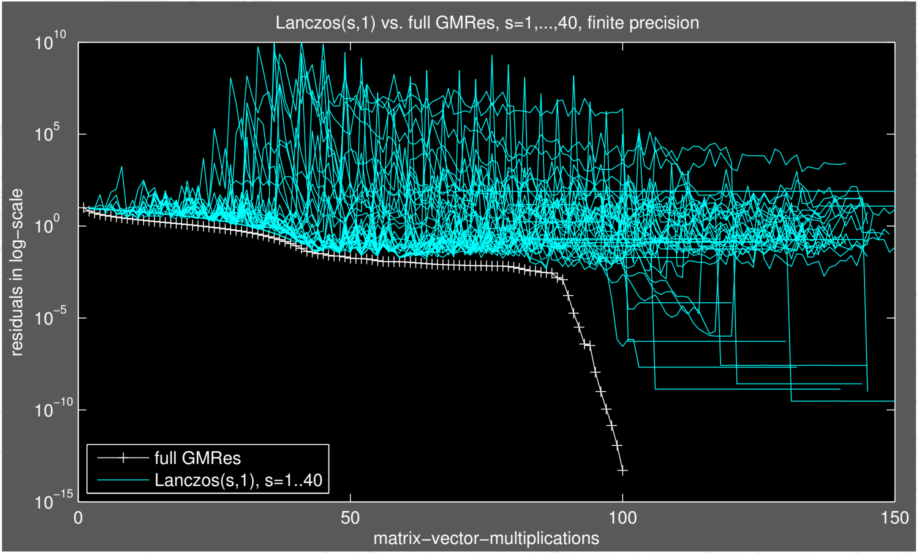
We remark that the prototype IDR algorithm suffered from instability for large values of s. We only consider new, stable implementations.

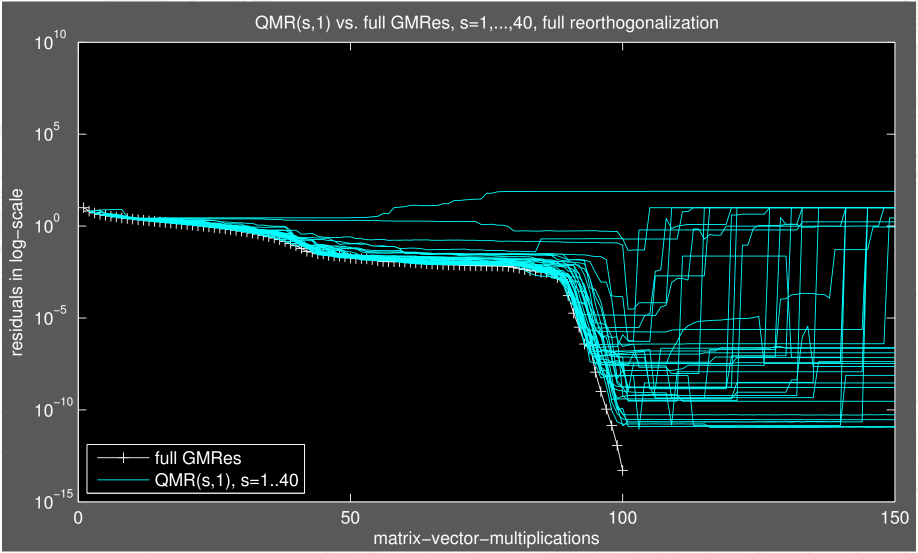
“Exact” Lanczos$(s, 1)$ versus full GMRES



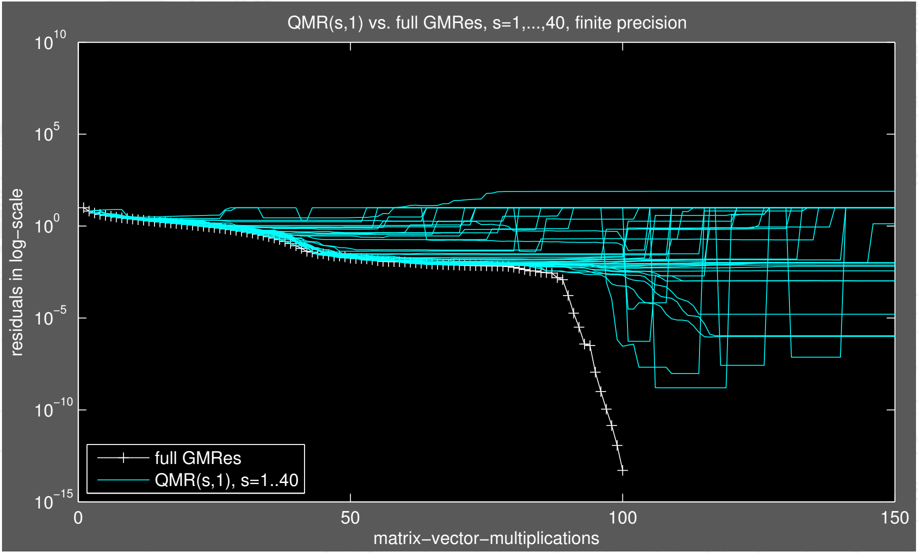
"Finite precision" Lanczos$(s, 1)$ versus full GMRES



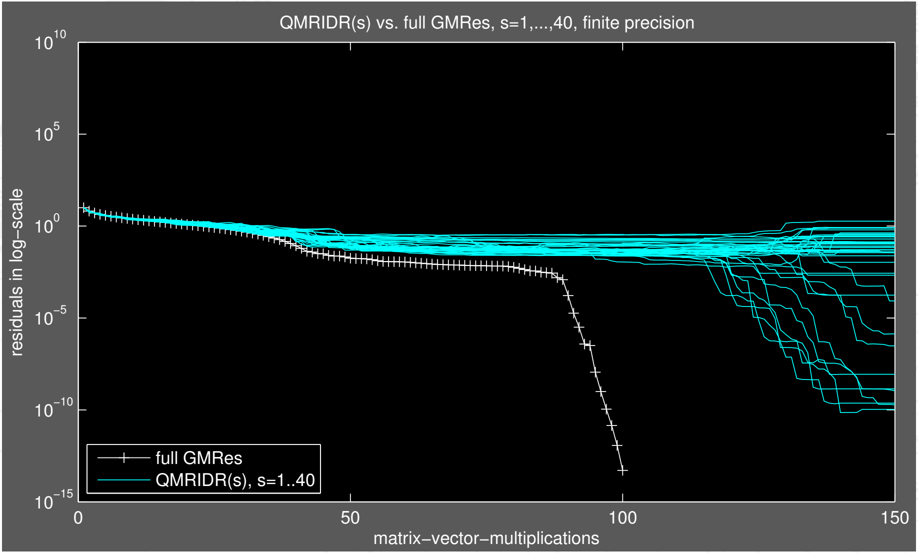
“Exact” QMR$(s, 1)$ versus full GMRES



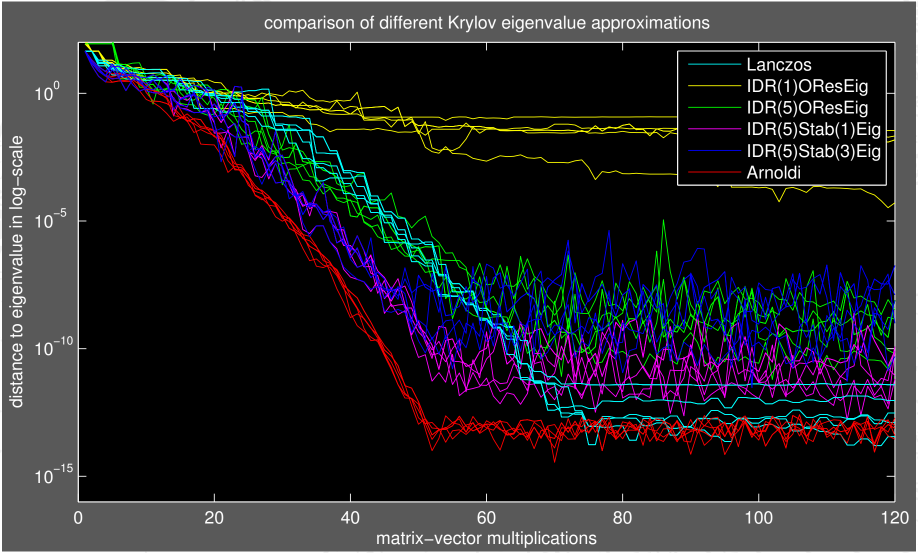
"Finite precision" QMR$(s, 1)$ versus full GMRes



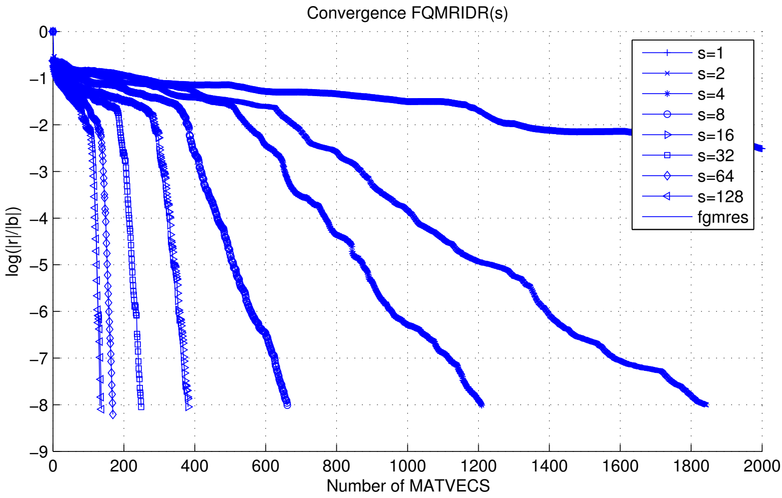
Finite precision QMRIDR(s) versus full GMRes



A comparison: IDR based eigenvalue solvers



Flexible QMRIDR(s)



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- An error analysis and a description of the finite precision behavior is desperately needed.

どうもありがとうございました。

Thank you very much for inviting me to 京都大学.

This talk is partially based on the following technical reports:

Eigenvalue computations based on IDR, Martin H. Gutknecht and Z., Bericht 145, Institut für Numerische Simulation, TUHH, 2010,

Flexible and multi-shift induced dimension reduction algorithms for solving large sparse linear systems, Martin B. van Gijzen, Gerard L.G. Sleijpen, and Z., Bericht 156, Institut für Numerische Simulation, TUHH, 2011.

An extended abstract can be found in the proceedings:

IDR versus other Krylov subspace solvers, Z., 2011.

Sonneveld, P. (2010).
On the convergence behaviour of IDR(s).
Technical Report 10-08, Department of Applied Mathematical Analysis, Delft University of Technology, Delft.