On the behavior of IDR(s) algorithms in finite precision

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Outline

Krylov methods in finite precision

. . . an introduction

IDR and IDR(s)

1976–1980: IDR

2006–2010: IDR(s)

IDR: two close relatives

IDR and Lanczos

IDR and Lanczos-type product methods

Numerical experiments

Many pictures — less mathematics

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Krylov subspace methods

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IDR and IDR(s) are Krylov subspace methods. The mth Krylov subspace $\mathcal{K}_m$ is defined for a given square matrix $\mathbf{A}$ and a starting vector $\mathbf{q}$ as follows,

$$\mathcal{K}_m(\mathbf{A}, \mathbf{q}) := \text{span}\{\mathbf{q}, \mathbf{A}\mathbf{q}, \ldots, \mathbf{A}^{m-1}\mathbf{q}\}.$$

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There is a natural isomorphism

$$\mathbf{v} \in \mathcal{K}_m \quad \Leftrightarrow \quad \mathbf{v} = \nu(\mathbf{A})\mathbf{q}$$

between vectors $\mathbf{v}$ in a Krylov subspace and polynomials $\nu \in \mathcal{P}_{m-1}$ (as long as the Krylov subspace $\mathcal{K}_m$ has full dimension $\dim(\mathcal{K}_m) = m$).

The origin of Krylov subspace methods

The Krylov matrices $\mathbf{K}_m := \left(\mathbf{q}, \mathbf{A}\mathbf{q}, \mathbf{A}^2\mathbf{q}, \ldots, \mathbf{A}^{m-1}\mathbf{q}\right)$ satisfy the matrix recurrence

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Suppose we choose upper triangular basis transformations $\mathbf{K}_m =: \mathbf{Q}_m\mathbf{R}_m$,

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Next we strip off the first column on both sides.

The connection to Hessenberg decompositions

The matrix $\underline{\mathbf{C}}_m \in \mathbb{C}^{(m+1)\times m}$ defined by

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We end up with a Hessenberg decomposition

$$\mathbf{A}\mathbf{Q}_m = \mathbf{Q}_{m+1}\underline{\mathbf{C}}_m =: \mathbf{Q}_m\mathbf{C}_m + \mathbf{q}_{m+1}\mathbf{c}_{m+1,m}\mathbf{e}_m^T, \tag{4}$$

where $\mathbf{C}_m$ is unreduced Hessenberg and measures the "ratio" of the basis transformations.

Classification of Krylov methods: Matrix based

There are three well-known approaches based on such Hessenberg decompositions (with $\|\mathbf{r}_0\|_2\mathbf{q}_1 = \mathbf{r}_0$), namely,

QOR: approximate $\mathbf{x} = \mathbf{A}^{-1}\mathbf{r}_0$ by $\mathbf{x}_m := \mathbf{Q}_m\mathbf{C}_m^{-1}\mathbf{e}_1\|\mathbf{r}_0\|_2$,

QMR: approximate $\mathbf{x} = \mathbf{A}^{-1}\mathbf{r}_0$ by $\underline{\mathbf{x}}_m := \mathbf{Q}_m\underline{\mathbf{C}}_m^{\dagger}\underline{\mathbf{e}}_1\|\mathbf{r}_0\|_2$,

Ritz-Galërkin: approximate part of $\mathbf{J} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ by $\mathbf{J}_m := \mathbf{S}_m^{-1}\mathbf{C}_m\mathbf{S}_m$
and part of $\mathbf{V}$ by $\mathbf{V}_m := \mathbf{Q}_m\mathbf{S}_m$, where $\mathbf{C}_m\mathbf{S}_m = \mathbf{S}_m\mathbf{J}_m$.

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It turns out to be helpful to look at the corresponding polynomial description: Krylov subspace methods compute elements from the polynomial Krylov subspace $\mathcal{K}_m$.

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The three classes of methods can be described using certain polynomials and polynomial interpolation:

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In finite precision the recurrence will only approximately be satisfied,

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Equations like Eqn. (5) will be called perturbed Hessenberg decompositions.

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In this talk, we restrict ourselves to $\mathcal{L}_m[z^{-1}]$, $\mathcal{L}_m[1-\delta_{z0}]$ and $\mathcal{R}_m$.

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- Generalization:

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- Generalization:

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Lagrange polynomials and QOR iterates

Theorem (The finite precision QOR iterates)

Suppose that all $\mathbf{C}_{\ell+1:m}$ *are regular. We define the* m*th QOR solution by*

$$\mathbf{z}_m := \mathbf{C}_m^{-1}\mathbf{e}_1\|\mathbf{r}_0\|_2$$

and the m*th QOR iterate by*

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Really sloppily speaking, in case of convergence,

$$\mathbf{x}_\infty = \mathbf{A}^{-1}\mathbf{r}_0 + \mathbf{A}^{-1}\mathbf{F}_\infty\mathbf{z}_\infty = \mathbf{A}^{-1}(\mathbf{r}_0 + \mathbf{F}_\infty\mathbf{z}_\infty).$$

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Proving convergence is the hard task.

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Two types of polynomials $\Rightarrow$ two expressions for the QOR residuals.

Residual polynomials and QOR residuals

Theorem (The finite precision QOR residuals)

Suppose that $\mathbf{q}_1 = \mathbf{r}_0/\|\mathbf{r}_0\|_2$ *and that all* $\mathbf{C}_{\ell+1:m}$ *are regular. Let* $\mathbf{x}_m$ *denote the* m*th QOR iterate and* $\mathbf{r}_m := \mathbf{r}_0 - \mathbf{A}\mathbf{x}_m$ *the corresponding residual.*

Then

$$\begin{aligned}\mathbf{r}_m &= \mathcal{R}_m(\mathbf{A})\mathbf{r}_0 + \sum_{\ell=1}^{m} \mathbf{z}_{\ell m}\mathcal{L}^0_{\ell+1:m}[1-\delta_{z0}](\mathbf{A})\mathbf{f}_\ell \\ &= \mathcal{R}_m(\mathbf{A})\mathbf{r}_0 - \sum_{\ell=1}^{m} \mathbf{z}_{\ell m}\mathcal{R}_{\ell+1:m}(\mathbf{A})\mathbf{f}_\ell + \mathbf{F}_m\mathbf{z}_m.\end{aligned} \tag{7}$$

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Thus, we need to understand how the Ritz values behave in finite precision.

Outline

Origin

In 1976 Sonneveld experimentally observed that for $\mathbf{B} \in \mathbb{C}^{n\times n}$ and a given starting vector $\mathbf{f}_0 \in \mathbb{C}^n$ and $\mathbf{f}_1 := \mathbf{B}\mathbf{f}_0$ the three-term recurrence

$$\mathbf{f}_{k+1} := \mathbf{B}(\mathbf{f}_k - \gamma_k(\mathbf{f}_k - \mathbf{f}_{k-1})), \quad \gamma_k := \frac{\mathbf{p}^{\mathsf{H}}\mathbf{f}_k}{\mathbf{p}^{\mathsf{H}}(\mathbf{f}_k - \mathbf{f}_{k-1})}$$

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He thus called this property “Induced Dimension Reduction” (IDR), and algorithms like the given three-term recurrence “IDR Algorithms”.

IDR Theorem

Sonneveld first made experiments and then gave a rigorous proof. It is easy to see that apart from the first two (arbitrarily chosen) residuals the constructed residuals are in the $\mathbf{B}$ image of the space $\mathcal{S} := \mathbf{p}^{\perp}$.

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Sonneveld proved that the dimensions of the spaces constructed are shrinking. This is the essence of the first IDR Theorem. He did not use the description as an orthogonal complement of a Krylov subspace as it is done here. We remark that generically $\dim(\mathcal{K}_n(\mathbf{B}^{-\mathsf{H}}, \mathbf{B}^{-\mathsf{H}}\mathbf{p})) = n$.

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Using the Krylov subspace point of view and the explicit orthogonalization against $\mathbf{p}$ before multiplication with $\mathbf{B}$, we see that indeed $\mathbf{f}_{2n} = \mathbf{B}\mathbf{o}_n = \mathbf{o}_n$.

IDR Algorithms

The three-term recurrence

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Another implementation by Sonneveld can be used to solve "genuine" linear systems. The idea is to rewrite the linear system to Richardson iteration form,

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad \mathbf{x} = (\mathbf{I} - \mathbf{A})\mathbf{x} + \mathbf{b} =: \mathbf{B}\,\mathbf{x} + \mathbf{b}.$$

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The classical Richardson iteration with a starting guess $\mathbf{x}_0$ is then given by

$$\mathbf{x}_{k+1} = (\mathbf{I} - \mathbf{A})\mathbf{x}_k + \mathbf{b}.$$

Primitive IDR

With $\mathbf{r}_0 := \mathbf{b} - \mathbf{A}\mathbf{x}_0$, the Richardson iteration is carried out as follows:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{r}_k, \quad \mathbf{r}_{k+1} = (\mathbf{I} - \mathbf{A})\mathbf{r}_k.$$

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The update of the iterates has to be modified accordingly,

$$\begin{aligned}
-\mathbf{A}(\mathbf{x}_{k+1} - \mathbf{x}_k) = \mathbf{r}_{k+1} - \mathbf{r}_k &= (\mathbf{I} - \mathbf{A})(\mathbf{r}_k + \gamma_k(\mathbf{r}_k - \mathbf{r}_{k-1}) - \mathbf{r}_k \\
&= (\mathbf{I} - \mathbf{A})(\mathbf{r}_k - \gamma_k\mathbf{A}(\mathbf{x}_k - \mathbf{x}_{k-1}) - \mathbf{r}_k \\
&= -\mathbf{A}(\mathbf{r}_k + \gamma_k(\mathbf{I} - \mathbf{A})(\mathbf{x}_k - \mathbf{x}_{k-1})) \\
\Leftrightarrow \quad \mathbf{x}_{k+1} - \mathbf{x}_k &= \mathbf{r}_k + \gamma_k(\mathbf{I} - \mathbf{A})(\mathbf{x}_k - \mathbf{x}_{k-1}) \\
&= \mathbf{r}_k + \gamma_k(\mathbf{x}_k - \mathbf{x}_{k-1} + \mathbf{r}_k - \mathbf{r}_{k-1}).
\end{aligned}$$

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Sonneveld terms the outcome the Primitive IDR Algorithm (Sonneveld, 2006):

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$$\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{r}_0$$
$$\mathbf{r}_1 = \mathbf{r}_0 - \mathbf{A}\mathbf{r}_0$$

For $k = 1, 2, \ldots$ do

$$\gamma_k = \mathbf{p}^{\mathsf{T}}\mathbf{r}_k / \mathbf{p}^{\mathsf{T}}(\mathbf{r}_{k-1} - \mathbf{r}_k)$$
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$$\mathbf{x}_{\text{old}} = \mathbf{x}_0$$
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While “not converged” do

$$\gamma = \mathbf{p}^{\mathsf{T}}\mathbf{r}_{\text{new}} / \mathbf{p}^{\mathsf{T}}(\mathbf{r}_{\text{old}} - \mathbf{r}_{\text{new}})$$
$$\mathbf{s} = \mathbf{r}_{\text{new}} + \gamma(\mathbf{r}_{\text{new}} - \mathbf{r}_{\text{old}})$$
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$$\mathbf{x}_{k+1} = \mathbf{x}_k + \gamma_k(\mathbf{x}_k - \mathbf{x}_{k-1}) + \mathbf{s}_k$$
$$\mathbf{r}_{k+1} = \mathbf{s}_k - \mathbf{A}\mathbf{s}_k$$

done

$$\mathbf{x}_{\text{old}} = \mathbf{x}_0$$
$$\mathbf{r}_{\text{old}} = \mathbf{b} - \mathbf{A}\mathbf{x}_{\text{old}}$$
$$\mathbf{x}_{\text{new}} = \mathbf{x}_{\text{old}} + \mathbf{r}_{\text{old}}$$
$$\mathbf{r}_{\text{new}} = \mathbf{r}_{\text{old}} - \mathbf{A}\mathbf{r}_{\text{old}}$$

While “not converged” do

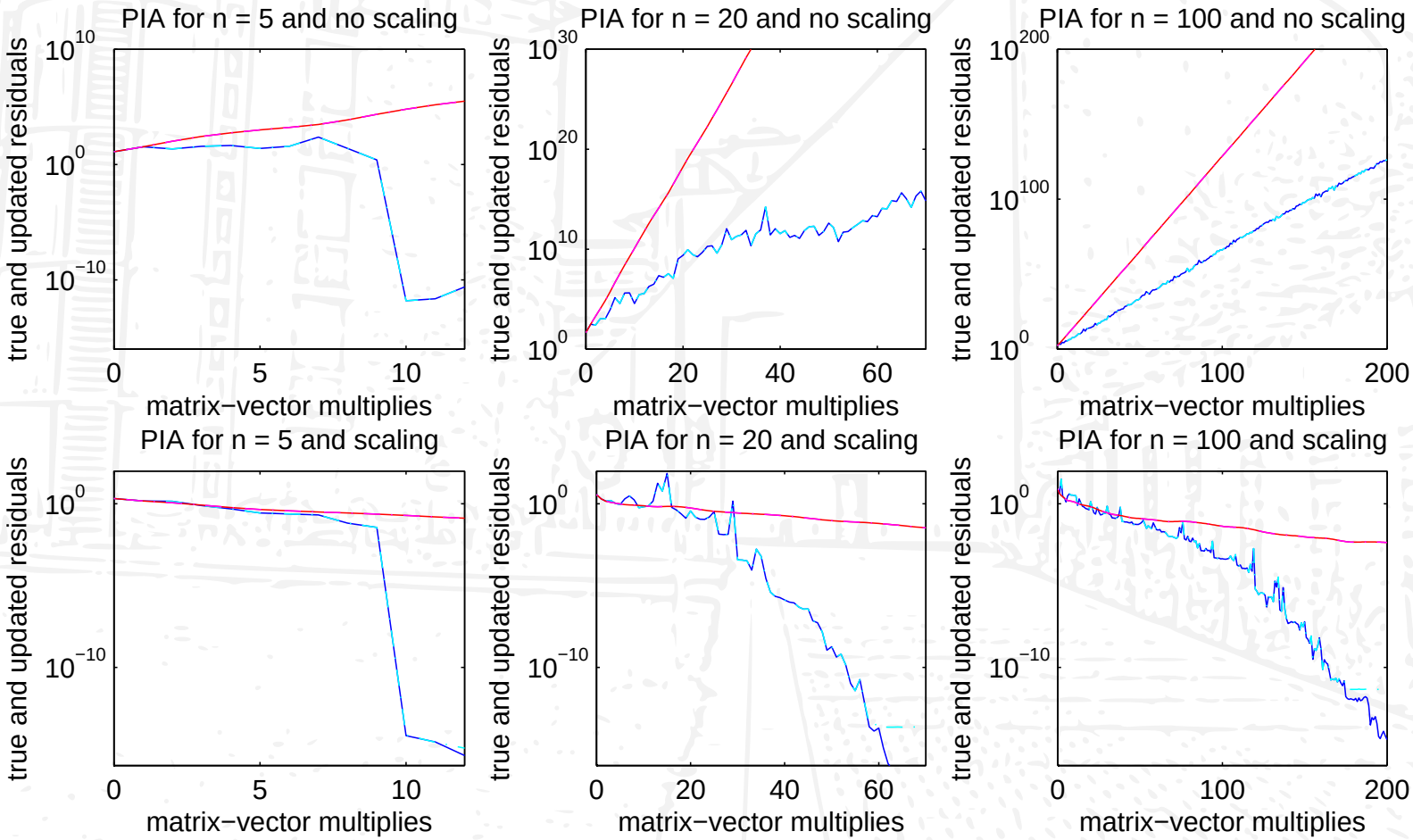
$$\gamma = \mathbf{p}^{\mathsf{T}}\mathbf{r}_{\text{new}} / \mathbf{p}^{\mathsf{T}}(\mathbf{r}_{\text{old}} - \mathbf{r}_{\text{new}})$$
$$\mathbf{s} = \mathbf{r}_{\text{new}} + \gamma(\mathbf{r}_{\text{new}} - \mathbf{r}_{\text{old}})$$
$$\mathbf{x}_{\text{tmp}} = \mathbf{x}_{\text{new}} + \gamma(\mathbf{x}_{\text{new}} - \mathbf{x}_{\text{old}}) + \mathbf{s}$$
$$\mathbf{r}_{\text{tmp}} = \mathbf{s} - \mathbf{A}\mathbf{s}$$
$$\mathbf{x}_{\text{old}} = \mathbf{x}_{\text{new}},\ \mathbf{x}_{\text{new}} = \mathbf{x}_{\text{tmp}}$$
$$\mathbf{r}_{\text{old}} = \mathbf{r}_{\text{new}},\ \mathbf{r}_{\text{new}} = \mathbf{r}_{\text{tmp}}$$

done

On the next slide we compare Richardson iteration (red) and PIA (blue).

Primitive IDR

Impressions of "finite termination" and acceleration in finite precision:



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Sonneveld never did use PIA, as he considered it to be too unstable, instead he went on with a corresponding acceleration of the Gauß-Seidel method. In (Sonneveld, 2008) he terms this method Accelerated Gauß-Seidel (AGS) and refers to it as “[t]he very first IDR-algorithm [..]”, see page 6, Ibid.

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In September 1979 Sonneveld did attend the IUTAM Symposium on Approximation Methods for Navier-Stokes Problems in Paderborn, Germany. At this symposium he presented a new variant of IDR based on a variable splitting $\mathbf{I} - \omega_j \mathbf{A}$, where ω_j is fixed for two steps and otherwise could be chosen freely, but non-zero.

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This algorithm with minimization of every second residual is included in the proceedings from 1980 (Wesseling and Sonneveld, 1980). The connection to Krylov methods, e.g., BiCG/Lanczos, is also given there.

Classical IDR

$\gamma_0 = 0$, $\mathbf{f}_0 = \mathbf{A}\mathbf{x}_0 - \mathbf{b}$, $\Delta\mathbf{g}_0 = \mathbf{o}_n$, $\Delta\mathbf{y}_0 = \mathbf{o}_n$
For $k = 1, \ldots$ do
  $\mathbf{s}_k = \mathbf{f}_{k-1} + \gamma_{k-1}\Delta\mathbf{g}_{k-1}$
  $\mathbf{t}_k = \mathbf{A}\mathbf{s}_k$
  if $k = 1$ or k is even
    $\omega_k = (\mathbf{t}_k^{\mathsf{H}}\mathbf{s}_k)/(\mathbf{t}_k^{\mathsf{H}}\mathbf{t}_k)$
  else
    $\omega_k = \omega_{k-1}$
  end
  $\Delta\mathbf{x}_k = \gamma_{k-1}\Delta\mathbf{y}_{k-1} - \omega_k\mathbf{s}_k$
  $\Delta\mathbf{f}_k = \gamma_{k-1}\Delta\mathbf{g}_{k-1} - \omega_k\mathbf{t}_k$
  $\mathbf{x}_k = \mathbf{x}_{k-1} + \Delta\mathbf{x}_k$
  $\mathbf{f}_k = \mathbf{f}_{k-1} + \Delta\mathbf{f}_k$
  if k is even
    $\Delta\mathbf{y}_k = \Delta\mathbf{y}_{k-1}$
    $\Delta\mathbf{g}_k = \Delta\mathbf{g}_{k-1}$
  else
    $\Delta\mathbf{y}_k = \Delta\mathbf{x}_k$
    $\Delta\mathbf{g}_k = \Delta\mathbf{f}_k$
  end
  $\gamma_k = -(\mathbf{p}^{\mathsf{H}}\mathbf{f}_k)/(\mathbf{p}^{\mathsf{H}}\Delta\mathbf{g}_k)$
done

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  if k is even
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  end
  $\gamma_k = -(\mathbf{p}^{\mathsf{H}}\mathbf{f}_k)/(\mathbf{p}^{\mathsf{H}}\Delta\mathbf{g}_k)$
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This is the original IDR Algorithm from page 551 of (Wesseling and Sonneveld, 1980).

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  if k is even
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  else
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This is the original IDR Algorithm from page 551 of (Wesseling and Sonneveld, 1980).

It uses OrthoRes(1) in the first step and a residual (these are the $-\mathbf{f}_{2j}$) minimization every second step.

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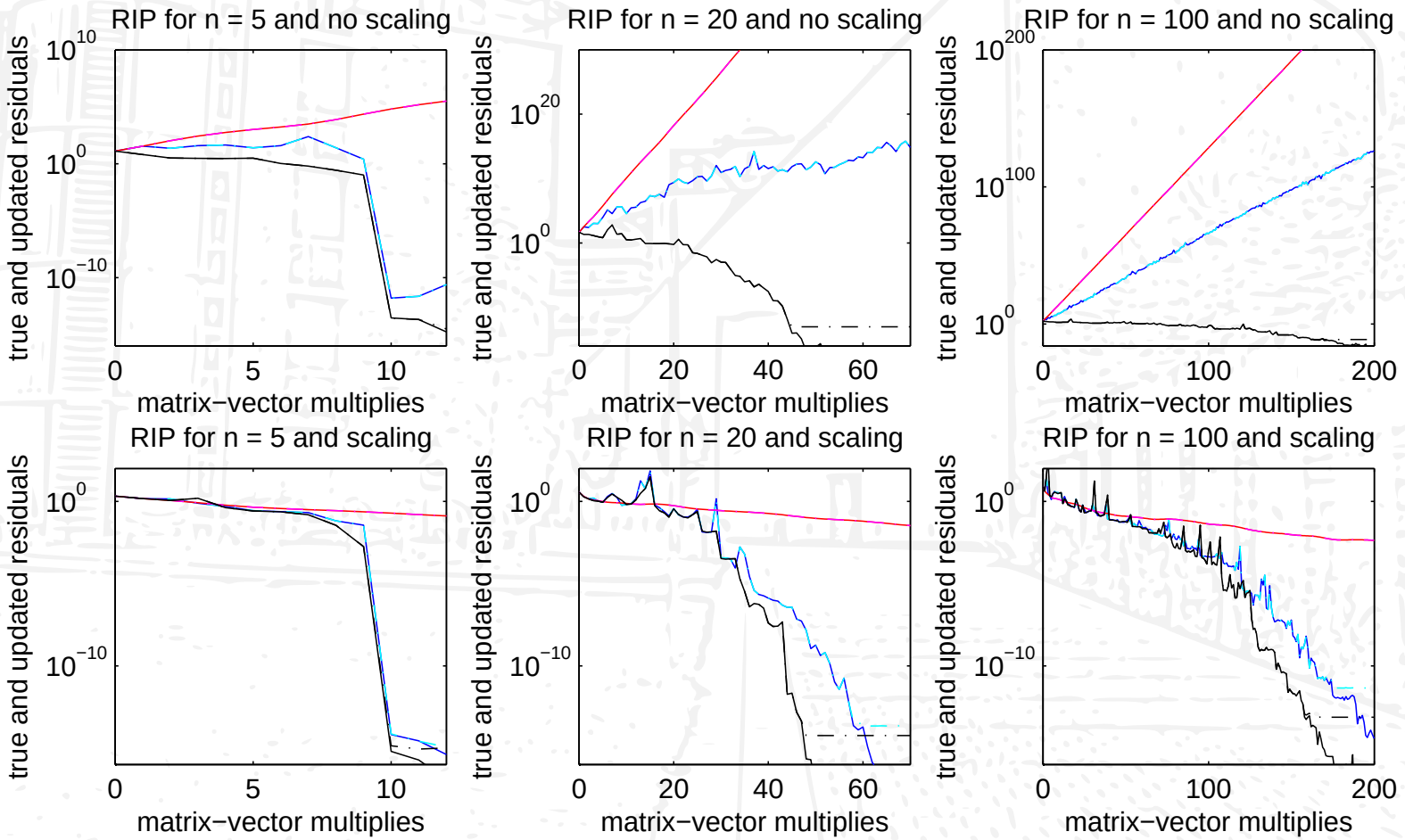
It uses OrthoRes(1) in the first step and a residual (these are the $-\mathbf{f}_{2j}$) minimization every second step.

The finite termination property follows from a generalization of the IDR Theorem based on commutativity of the linear polynomials $\mathbf{I} - \omega_j\mathbf{A}$.

Classical IDR

A numerical comparison of Richardson iteration, original IDR, and PIA.



Brothers of classical and primitive IDR

In 1976 Sonneveld considered the acceleration of Gauß-Seidel (AGS). Similarly, the IDR philisophy can be used as an accelerator for the other classical splitting methods like Jacobi, SOR, SSOR, or Chebyshev, or even more general semi-iterative methods.

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The numerical behavior is not very promising. But this picture changes, when we use more shadow vectors . . .

Outline

Rebirth of IDR: IDR(s)

In 2006, 30 years after inventing IDR, Sonneveld together with van Gijzen reconsidered IDR and came up with a variant called IDR(s) that used orthogonalization against a larger space, where s denotes the dimension of that space.

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Algorithmically, the transition from IDR to IDR(s) corresponds to replacing the single vector $\mathbf{p} \in \mathbb{C}^n$ with a matrix $\mathbf{P} \in \mathbb{C}^{n \times s}$, $1 \leqslant s \leqslant n$.

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To analyze IDR and IDR(s), we have to consider generalized Hessenberg decompositions (also referred to as rational Hessenberg decompositions) and to generalize QOR, QMR and Ritz-Galërkin.

We have to prove that the expressions for the iterates and residuals based on polynomials are still valid. But: All these approaches extend easily to generalized Hessenberg decompositions.

The prototype IDR(s) (without the recurrences for $\mathbf{x}_n$ and thus already slightly rewritten)

$\mathbf{r}_0 = \mathbf{b} - \mathbf{A}\mathbf{x}_0$
compute $\mathbf{R}_{s+1} = \mathbf{R}_{0:s} = (\mathbf{r}_0, \ldots, \mathbf{r}_s)$ using, e.g., ORTHORES
$\nabla\mathbf{R}_{1:s} = (\nabla\mathbf{r}_1, \ldots, \nabla\mathbf{r}_s) = (\mathbf{r}_1 - \mathbf{r}_0, \ldots, \mathbf{r}_s - \mathbf{r}_{s-1})$
$n \leftarrow s+1, j \leftarrow 1$
while not converged
  $\mathbf{c}_n = (\mathbf{P}^{\mathsf{H}}\nabla\mathbf{R}_{n-s:n-1})^{-1}\mathbf{P}^{\mathsf{H}}\mathbf{r}_{n-1}$
  $\mathbf{v}_{n-1} = \mathbf{r}_{n-1} - \nabla\mathbf{R}_{n-s:n-1}\mathbf{c}_n$
  compute ω_j
  $\nabla\mathbf{r}_n = -\nabla\mathbf{R}_{n-s:n-1}\mathbf{c}_n - \omega_j\mathbf{A}\mathbf{v}_{n-1}$
  $\mathbf{r}_n = \mathbf{r}_{n-1} + \nabla\mathbf{r}_n, n \leftarrow n+1$
  $\nabla\mathbf{R}_{n-s:n-1} = (\nabla\mathbf{r}_{n-s}, \ldots, \nabla\mathbf{r}_{n-1})$
  for $k = 1, \ldots, s$
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A few remarks:

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A few remarks:

We can start with any (simple) Krylov subspace method.

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IDR(s)ORes is based on oblique projections and $s+1$ consecutive multiplications with the same linear factor $\mathbf{I} - \omega_j \mathbf{A}$.

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We already noted that essential features of Krylov subspace methods can be described by a Hessenberg decomposition

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$$\mathbf{A}\mathbf{Q}_n + \mathbf{F}_n = \mathbf{Q}_{n+1}\underline{\mathbf{H}}_n = \mathbf{Q}_n\mathbf{H}_n + \mathbf{q}_{n+1}h_{n+1,n}\mathbf{e}_n^T. \tag{9}$$

The matrix $\mathbf{H}_n$ of the perturbed variant will, in general, still be unreduced.

IDR: Generalized Hessenberg decompositions

In case of IDR, we have to consider generalized Hessenberg decompositions

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Generalized Hessenberg decompositions correspond to an oblique projection of the pencil $(\mathbf{A}, \mathbf{I})$ to the pencil $(\mathbf{H}_n, \mathbf{U}_n)$ as long as $\mathbf{Q}_{n+1}$ has full rank,

$$\begin{aligned}\widehat{\mathbf{Q}}_n^H(\mathbf{A}, \mathbf{I})\mathbf{Q}_n\mathbf{U}_n &= \widehat{\mathbf{Q}}_n^H(\mathbf{A}\mathbf{Q}_n\mathbf{U}_n, \mathbf{Q}_n\mathbf{U}_n) \\ &= \widehat{\mathbf{Q}}_n^H(\mathbf{Q}_{n+1}\underline{\mathbf{H}}_n, \mathbf{Q}_n\mathbf{U}_n) = (\underline{\mathbf{I}}_n^T\underline{\mathbf{H}}_n, \mathbf{U}_n) = (\mathbf{H}_n, \mathbf{U}_n),\end{aligned} \tag{12}$$

where $\widehat{\mathbf{Q}}_n^H := \underline{\mathbf{I}}_n^T\mathbf{Q}_{n+1}^\dagger$.

Understanding IDR: OrthoRes-type methods

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The IDR recurrences of IDR(s)ORes can be summarized by

$$\begin{aligned}
\mathbf{v}_{n-1} &:= \mathbf{r}_{n-1} - \nabla\mathbf{R}_{n-s:n-1}\mathbf{c}_n = \mathbf{R}_{n-s-1:n-1}\mathbf{y}_n \\
&= \left(1-\gamma_s^{(n)}\right)\mathbf{r}_{n-1} + \textstyle\sum_{\ell=1}^{s-1}\left(\gamma_{s-\ell+1}^{(n)} - \gamma_{s-\ell}^{(n)}\right)\mathbf{r}_{n-\ell-1} + \gamma_1^{(n)}\,\mathbf{r}_{n-s-1}\,, \\
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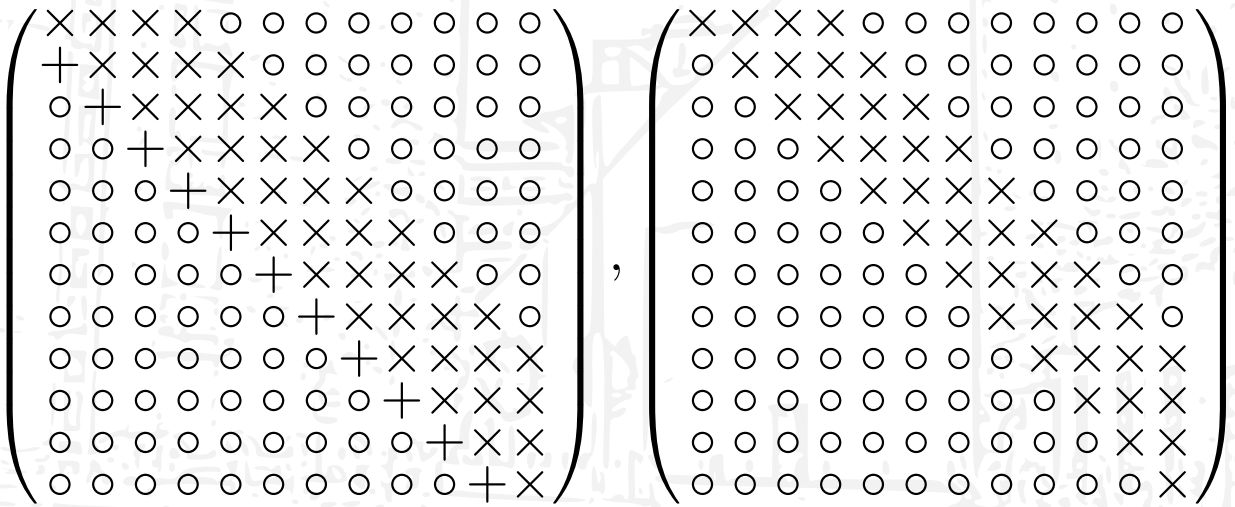
The IDR(s)ORes pencil, the so-called Sonneveld pencil $(\mathbf{Y}_n^\circ, \mathbf{Y}_n\mathbf{D}_\omega^{(n)})$, can be depicted by



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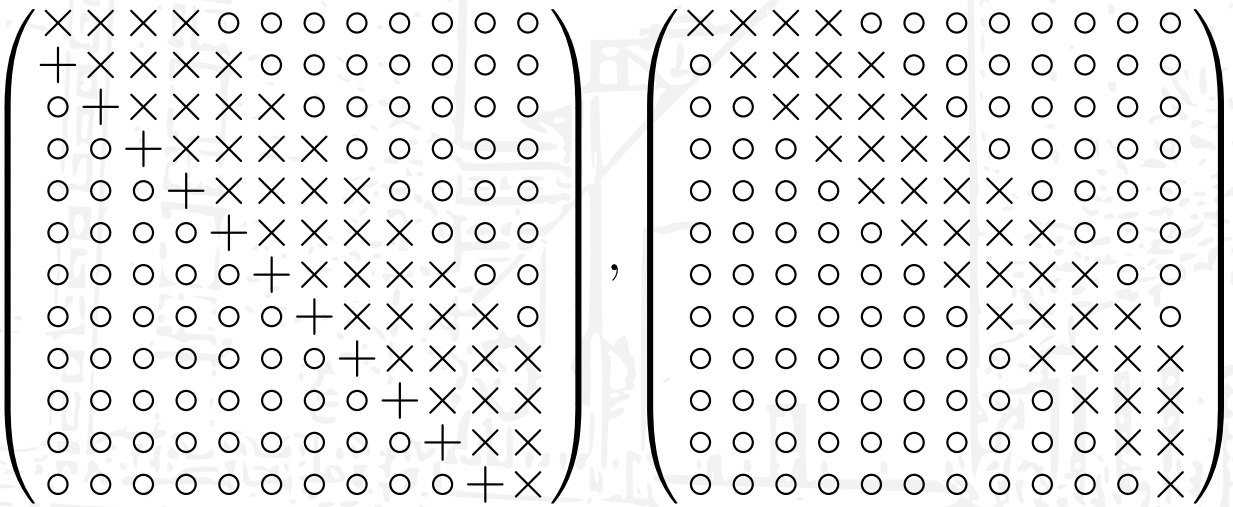

The upper triangular matrix $\mathbf{Y}_n\mathbf{D}_\omega^{(n)}$ could be inverted, which results in the Sonneveld matrix, a full unreduced Hessenberg matrix.

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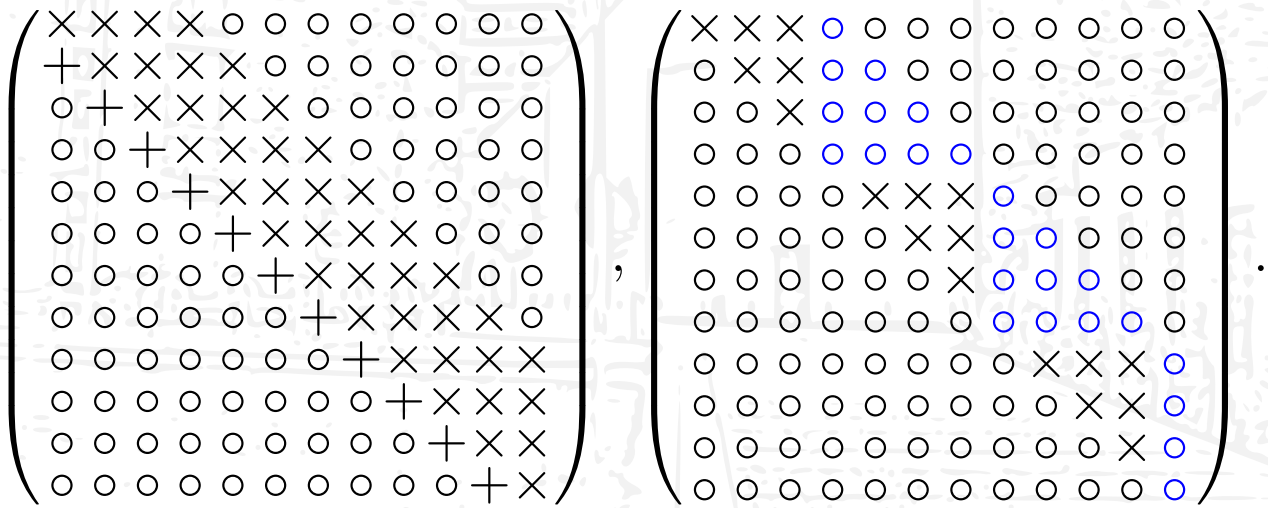

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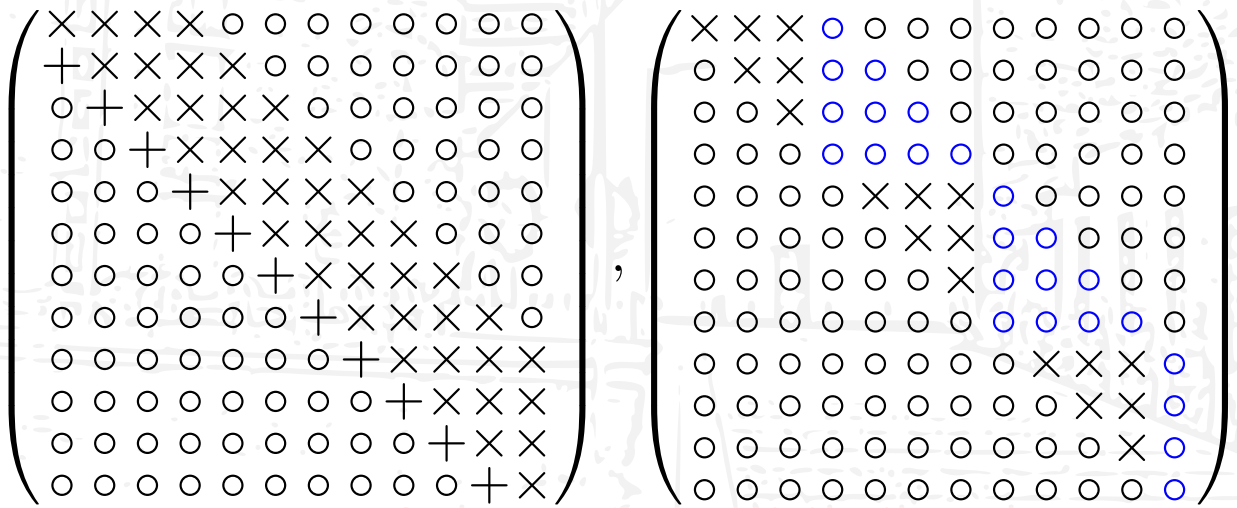
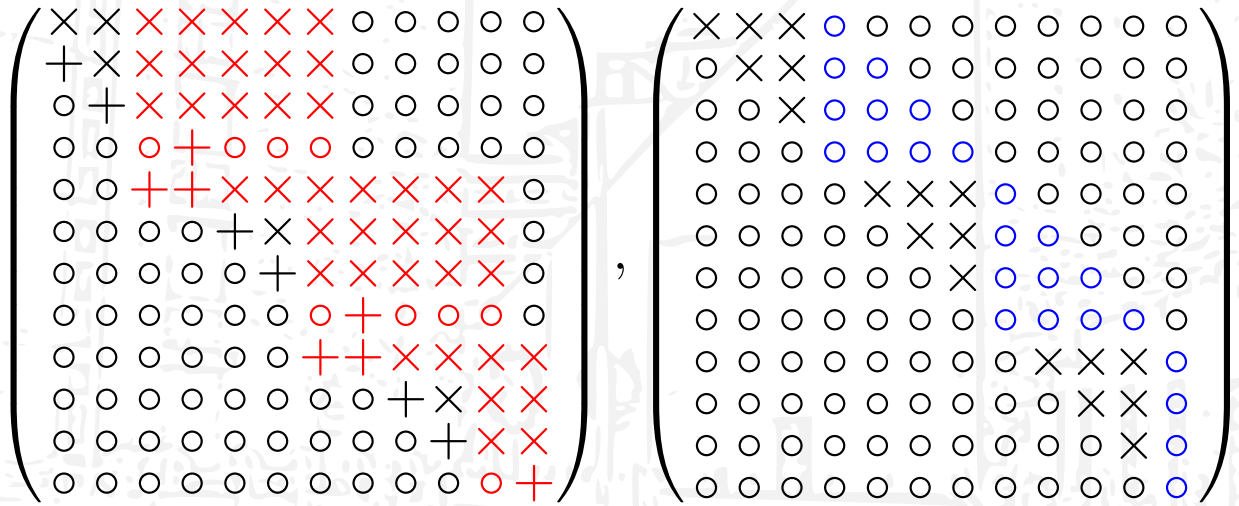


We get rid of the infinite eigenvalues using a change of basis (Gauß/Schur).

Understanding IDR: Gaussian elimination

The deflated purified IDR(s)ORes pencil, after the elimination step $(\mathbf{Y}_n^{\circ}\mathbf{G}_n, \mathbf{U}_n\mathbf{D}_{\omega}^{(n)})$, can be depicted by



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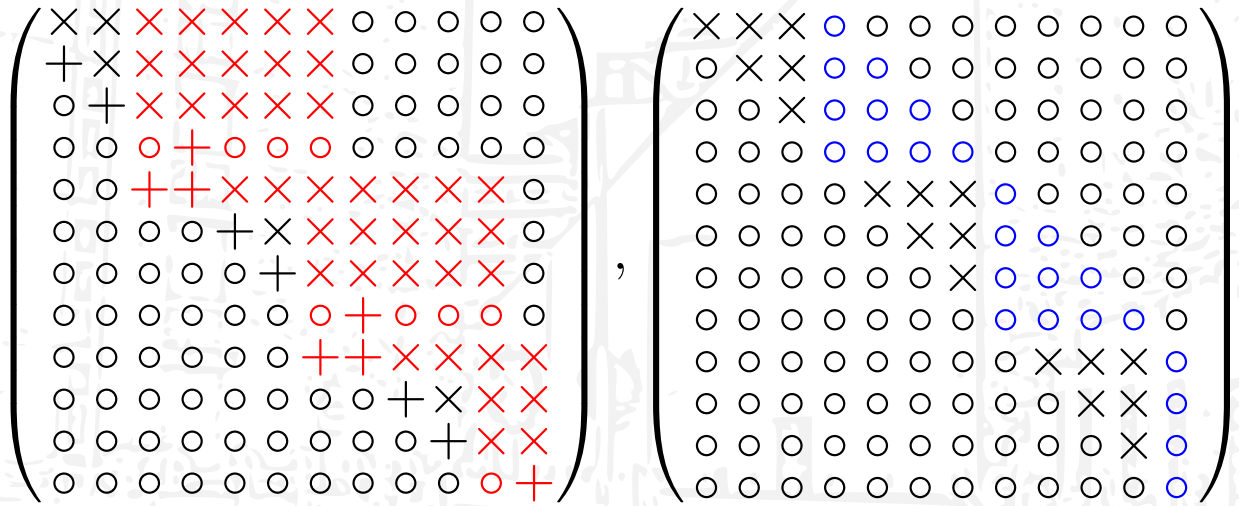

Using Laplace expansion of the determinant of $z\mathbf{U}_n\mathbf{D}_\omega^{(n)} - \mathbf{Y}_n^\circ\mathbf{G}_n$ we can get rid of the trivial constant factors corresponding to infinite eigenvalues. This amounts to a deflation.

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Let D denote an deflation operator that removes every $s+1$th column and row from the matrix the operator is applied to.

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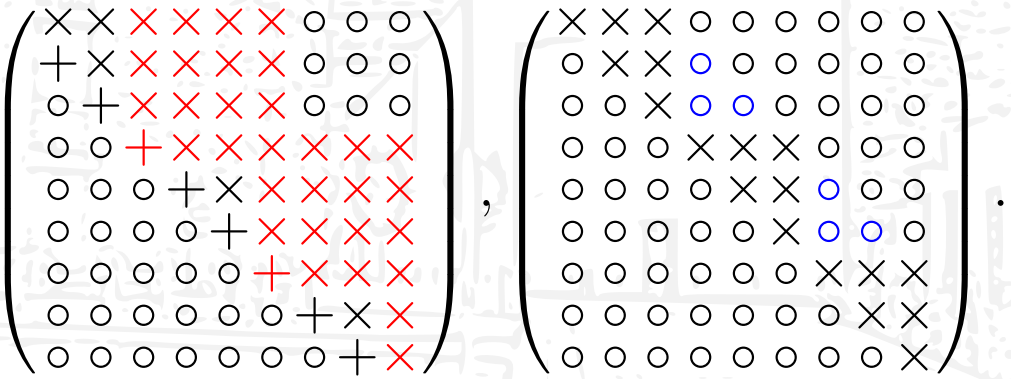

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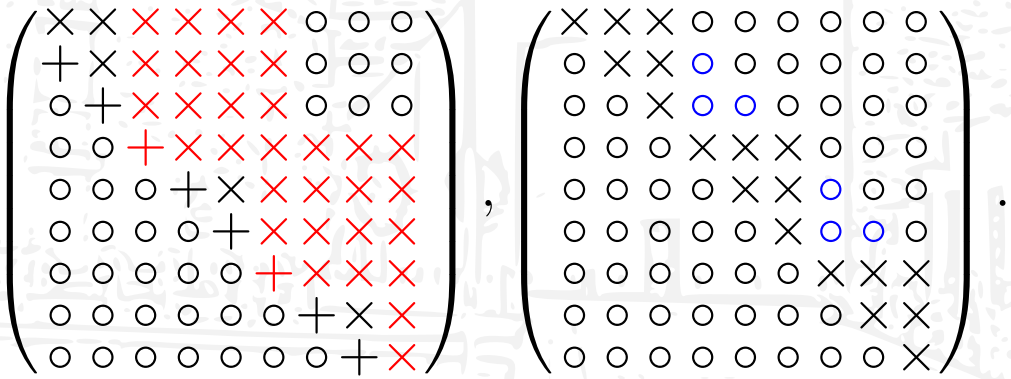

The block-diagonal matrix $D(\mathbf{U}_n \mathbf{D}_\omega^{(n)})$ has invertible upper triangular blocks and can be inverted to expose the underlying Lanczos process.

IDR: a Lanczos process with multiple left-hand sides

Inverting the block-diagonal matrix $D(\mathbf{U}_n\mathbf{D}_\omega^{(n)}))$ gives an algebraic eigenvalue problem with a block-tridiagonal unreduced upper Hessenberg matrix

$$\mathbf{L}_n := D(\mathbf{Y}_n^\circ\mathbf{G}_n)\cdot D(\mathbf{U}_n\mathbf{D}_\omega^{(n)}))^{-1} = \begin{pmatrix} \times & \times & \times & \times & \times & \times & \circ & \circ & \circ \\ + & \times & \times & \times & \times & \times & \circ & \circ & \circ \\ \circ & + & \times & \times & \times & \times & \circ & \circ & \circ \\ \circ & \circ & + & \times & \times & \times & \times & \times & \times \\ \circ & \circ & \circ & + & \times & \times & \times & \times & \times \\ \circ & \circ & \circ & \circ & + & \times & \times & \times & \times \\ \circ & \circ & \circ & \circ & \circ & + & \times & \times & \times \\ \circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times \end{pmatrix}.$$

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The extended matrix version $\underline{\mathbf{L}}_n$ satisfies

$$\mathbf{A}\mathbf{Q}_n = \mathbf{Q}_{n+1}\underline{\mathbf{L}}_n,$$

where the reduced residuals $\mathbf{q}_{js+k}$, $k = 0, \ldots, s-1$, $j = 0, 1, \ldots$, with $\Omega_0(z) \equiv 1$ and $\Omega_j(z) = \prod_{k=1}^{j}(1-\omega_k z)$ are given by $\Omega_j(\mathbf{A})\mathbf{q}_{js+k} = \mathbf{r}_{j(s+1)+k}$.

IDR: a Lanczos process with multiple left-hand sides

The reduced residuals are defined by

$$\Omega_j(\mathbf{A})\mathbf{q}_{js+k} = \mathbf{r}_{j(s+1)+k} = (\mathbf{I} - \omega_j\mathbf{A})\mathbf{v}_{j(s+1)+k-1}$$

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Using induction (Sleijpen et al., 2008) one can prove that $\mathbf{q}_{js+k} \perp \mathcal{K}_j(\mathbf{A}^H, \mathbf{P})$; thus, this is a two-sided Lanczos process with s left and one right starting vectors.

IDR: a Lanczos process with multiple left-hand sides

The reduced residuals are defined by

$$\Omega_j(\mathbf{A})\mathbf{q}_{js+k} = \mathbf{r}_{j(s+1)+k} = (\mathbf{I} - \omega_j\mathbf{A})\mathbf{v}_{j(s+1)+k-1}$$

and every $\mathbf{v}_{j(s+1)+k-1}$ is orthogonal to $\mathbf{P}$. Thus, $\mathbf{q}_{js+k} \perp \Omega_{j-1}(\mathbf{A}^{\mathsf{H}})\mathbf{P}$.

Using induction (Sleijpen et al., 2008) one can prove that $\mathbf{q}_{js+k} \perp \mathcal{K}_j(\mathbf{A}^{\mathsf{H}}, \mathbf{P})$; thus, this is a two-sided Lanczos process with s left and one right starting vectors.

This can more easily be proven using the representations ($\mathcal{S} := \mathbf{P}^{\perp}$)

$$\mathcal{G}_0 = \mathcal{K}(\mathbf{A}, \mathbf{r}_0), \quad \text{where } \mathcal{K}(\mathbf{A}, \mathbf{r}_0) \text{ denotes the } \textit{full} \text{ Krylov subspace,}$$

$$\mathcal{G}_j = \bigcap_{k=0}^{j-1} \Omega_k(\mathbf{A})^{-1}\Omega_j(\mathbf{A})(\mathcal{S}) = \Big(\mathop{+}_{k=0}^{j-1} \Omega_j(\mathbf{A})^{-\mathsf{H}}\Omega_k(\mathbf{A})^{\mathsf{H}}\{\mathbf{P}\}\Big)^{\perp}$$

$$= \Big(\Omega_j(\mathbf{A})^{-\mathsf{H}}\mathcal{K}_j(\mathbf{A}^{\mathsf{H}}, \mathbf{P})\Big)^{\perp} = \Omega_j(\mathbf{A})\Big(\mathcal{K}_j(\mathbf{A}^{\mathsf{H}}, \mathbf{P})\Big)^{\perp}$$

of the Sonneveld spaces.

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This has to be compared with Theorem 4.2 in (Sleijpen et al., 2008) and with Theorem 4.1 in (Simoncini and Szyld, 2009) (similar result; slightly different method of proof).

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The first equality

$$\mathcal{G}_j = \bigcap_{k=0}^{j-1} \Omega_k(\mathbf{A})^{-1}\Omega_j(\mathbf{A})(\mathcal{S}) = \bigcap_{k=1}^{j} (\mathbf{I} - \omega_j\mathbf{A}) \cdots (\mathbf{I} - \omega_k\mathbf{A})(\mathcal{S})$$

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follows from the observations that

- the first $s+1$ residuals obviously are in $\mathcal{G}_0 := \mathcal{K}(\mathbf{A}, \mathbf{r}_0)$,
- the next $s+1$ residuals (or any other vectors in $\mathcal{G}_1$) are in the $\mathbf{I} - \omega_1\mathbf{A}$ image of $\mathcal{S} = \mathbf{P}^{\perp}$,
- the last $s+1$ residuals are in the $\mathbf{I} - \omega_j\mathbf{A}$ image of $\mathcal{S} = \mathbf{P}^{\perp}$,
- the last residuals are $\mathbf{I} - \omega_j\mathbf{A}$ images of linear combinations of previously obtained images $(\mathbf{I} - \omega_{j-1}\mathbf{A}) \cdots (\mathbf{I} - \omega_k\mathbf{A})$ of $\mathcal{S} = \mathbf{P}^{\perp}$.

IDR: a Lanczos process with multiple left-hand sides

The second equality

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The second relations are basic linear algebra. The first relation follows from

$$\mathbf{P}^{\perp} = \left\{\mathbf{v} \in \mathbb{C}^n \mid \mathbf{P}^{\mathsf{H}}\mathbf{v} = \mathbf{o}_n\right\} \quad \Rightarrow \quad \mathbf{B}\mathbf{P}^{\perp} = \left\{\mathbf{B}\mathbf{v} \in \mathbb{C}^n \mid \mathbf{P}^{\mathsf{H}}\mathbf{v} = \mathbf{o}_n\right\},$$

since, for invertible $\mathbf{B}$,

$$\mathbf{y} \in \mathbf{B}\mathbf{P}^{\perp} \;\Leftrightarrow\; \left\{\mathbf{y} = \mathbf{B}\mathbf{v} \wedge \mathbf{P}^{\mathsf{H}}\mathbf{v} = \mathbf{o}_n\right\} \;\Leftrightarrow\; \mathbf{P}^{\mathsf{H}}\mathbf{v} = \mathbf{P}^{\mathsf{H}}\mathbf{B}^{-1}\mathbf{y} = (\mathbf{B}^{-\mathsf{H}}\mathbf{P})^{\mathsf{H}}\mathbf{y} = \mathbf{o}_n.$$

IDR: a Lanczos process with multiple left-hand sides

The third and fourth equality

$$\left(\mathop{+}_{k=0}^{j-1} \Omega_j(\mathbf{A})^{-\mathsf{H}} \Omega_k(\mathbf{A})^{\mathsf{H}} \{\mathbf{P}\} \right)^{\perp} = \left(\Omega_j(\mathbf{A})^{-\mathsf{H}} \mathcal{K}_j(\mathbf{A}^{\mathsf{H}}, \mathbf{P}) \right)^{\perp}$$

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are satisfied

- since the polynomials $\Omega_k(\mathbf{A})$, $0 \leqslant k < j$ form a basis of the space of polynomials of degree less j, and
- by the property proved on the last slide, respectively.

Generalizations of IDR(s)

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The new vectors $\mathbf{v}_n$ and $\mathbf{r}_{n+1}$ are in this general setting given by the updates

$$\mathbf{v}_n = \mathbf{r}_n - \sum_{i=1}^{s} \mathbf{g}_{n-i}\gamma_i =: \mathbf{r}_n - \mathbf{G}_n\mathbf{c}_n, \quad \text{and thus,}$$

$$\mathbf{r}_{n+1} = (\mathbf{I} - \omega\mathbf{A})\mathbf{v}_n = \mathbf{r}_n - \omega\mathbf{A}\mathbf{v}_n - \sum_{i=1}^{s} \mathbf{g}_{n-i}\gamma_i,$$

where $\mathbf{c}_n$ is determined such that $\mathbf{P}^{\mathsf{H}}\mathbf{v}_n = \mathbf{o}$.

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- In (Sleijpen and Abe, 2010) the ideas behind BiCGStab2 (Gutknecht, 1993) and GPBiCG (Zhang, 1997) are considered.

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The minimization used in IDR(s)ORes and IDR(s)BiO results in values ω_j which are in the field of values of $\mathbf{A}^{-\mathsf{H}}$, thus Simoncini and Szyld suggest to use a few steps of the Arnoldi method to compute some Ritz values, which are then used in some ordering as $1/\omega_j$ values.

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For real nonsymmetric matrices this typically results in an algorithm based on complex arithmetic in place of real arithmetic.

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Last but not least: Certain old ideas have been reactivated. Sonneveld presented the hitherto unpublished Accelerated Gauß-Seidel (AGS) method at the Kyoto Forum on Krylov Subspace Methods in 2008.

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- IDR(s)-Jacobi (w/o adaptive tuning),
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These approaches result in a “tight packing” of preconditioning and Krylov subspace methods, compare with PIA. In most of these methods the ω_j are fixed by the splitting chosen.

Outline

Similarities between Lanczos and IDR(1)

We have seen that IDR(s) is a Lanczos-type product method (LTPM), where the underlying Lanczos process uses s left and one right starting vectors.

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The only difference between the Lanczos process underlying IDR(1) and the finite precision Lanczos method is the perturbation term. Yet we will see that the structural constraints result in totally different deviations from the theoretical behavior of the Lanczos method.

Chris Paige, Anne Greenbaum, the Lanczos process

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Thus far, this is maybe the only successful error analysis ever carried out for a perturbed short-term Krylov subspace method.

Lanczos' method in finite precision

We used the diagonal matrix

$$\mathbf{A} = \texttt{diag([linspace(0,1,50),3])}$$

and the starting vector

$$\mathbf{e} = \texttt{ones(51,1)}$$

in an implementation of Lanczos' method in MATLAB on a PC conforming to ANSI/IEEE 754 with machine precision $\texttt{eps(1)} = 2^{-52} \approx 2.2204 \cdot 10^{-16}$.

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Eigenvalues and eigenvectors are computed using MRRR, i.e., LAPACK's routine `DSTEGR`, since MATLAB's `eig` (using LAPACK's `DSYEV`, i.e., the QR algorithm implemented as `DSTEQR`) fails in delivering accurate eigenvectors.

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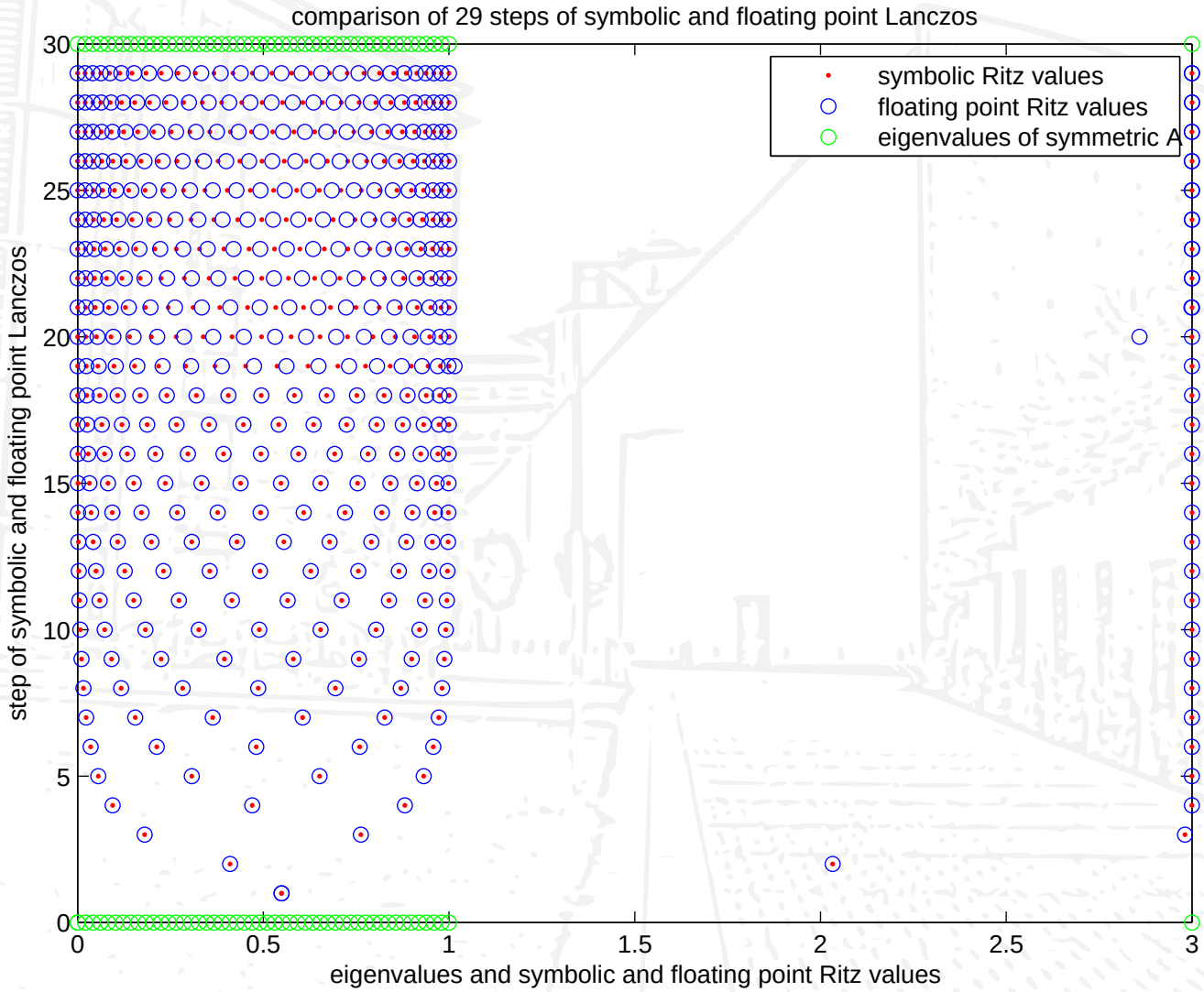

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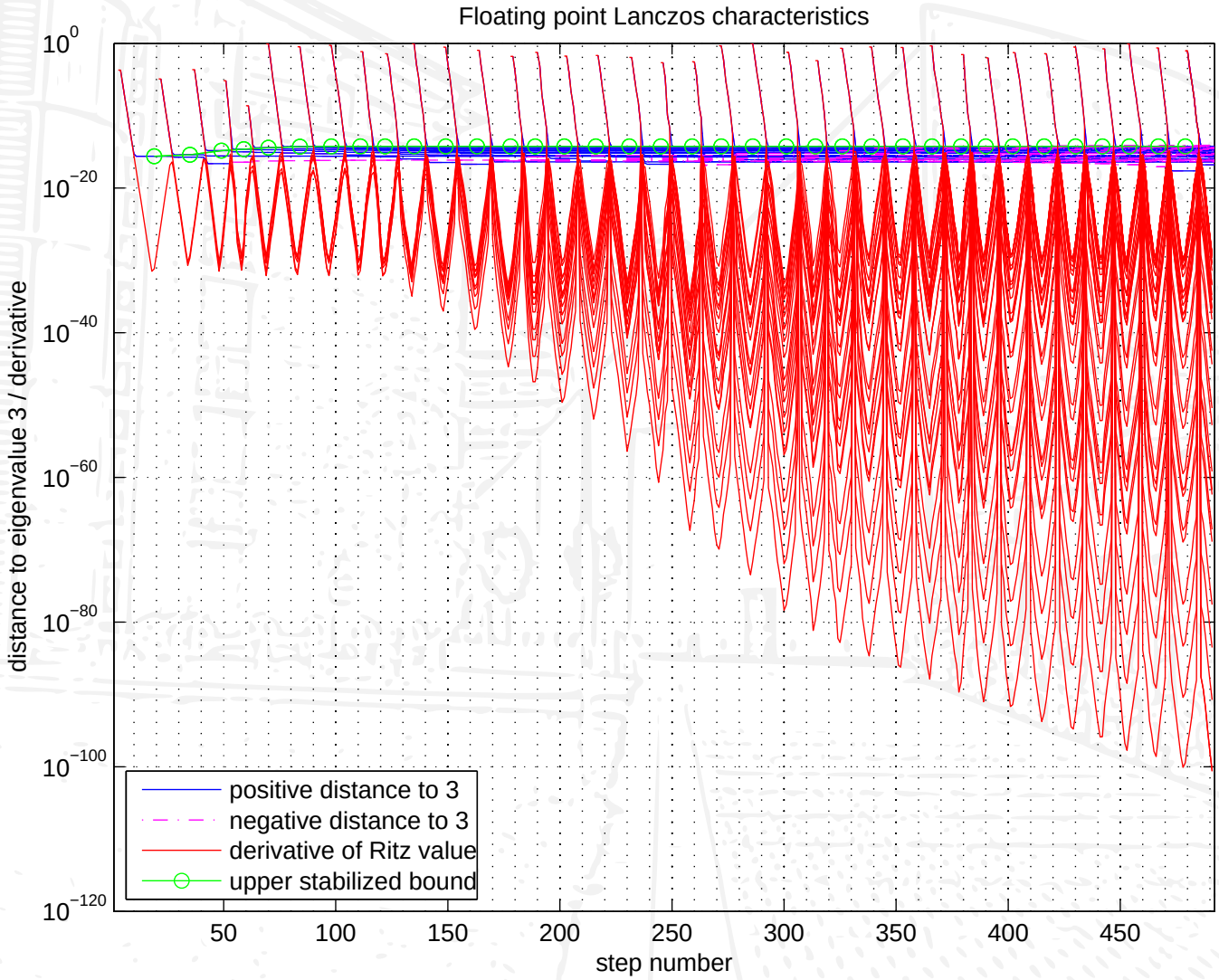

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We note that we can observe multiple copies, but this time the approximation quality is reduced after a couple of steps, all Ritz values computed after certain steps show worse behavior than before.

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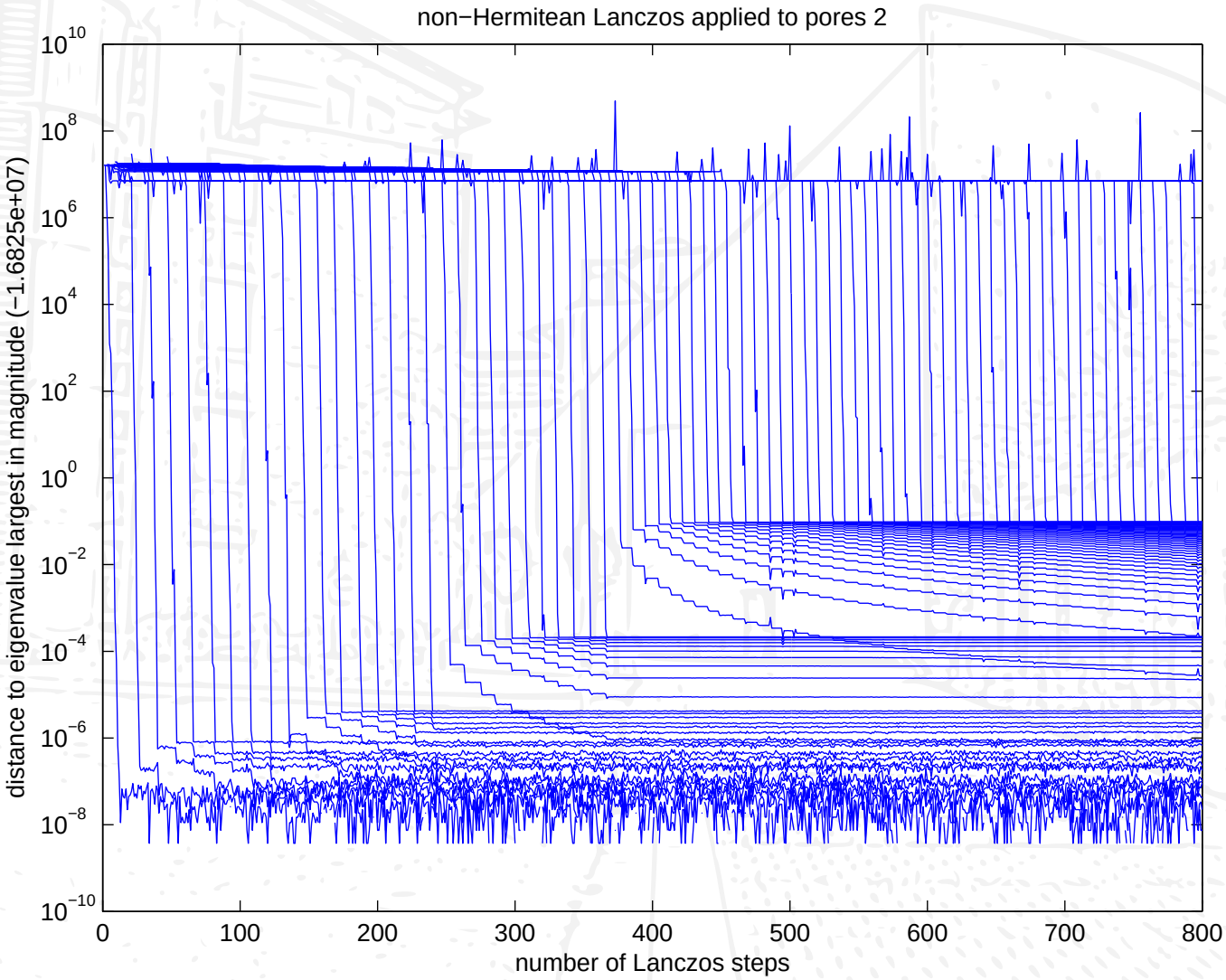

Outline

IDR, IDR(1), and BiCGStab

We already noted that IDR, IDR(1), and BiCGStab are in some sense equivalent: They produce every second step the same residual. These steps are related to the Lanczos method and the minimizers ω_j.

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The eigenvalue approximations corresponding to BiCGStab are easier to analyze than those corresponding to IDR or IDR(1), since in BiCGStab the coefficients of the tridiagonal matrix of the underlying Lanczos method are explicitely computed.

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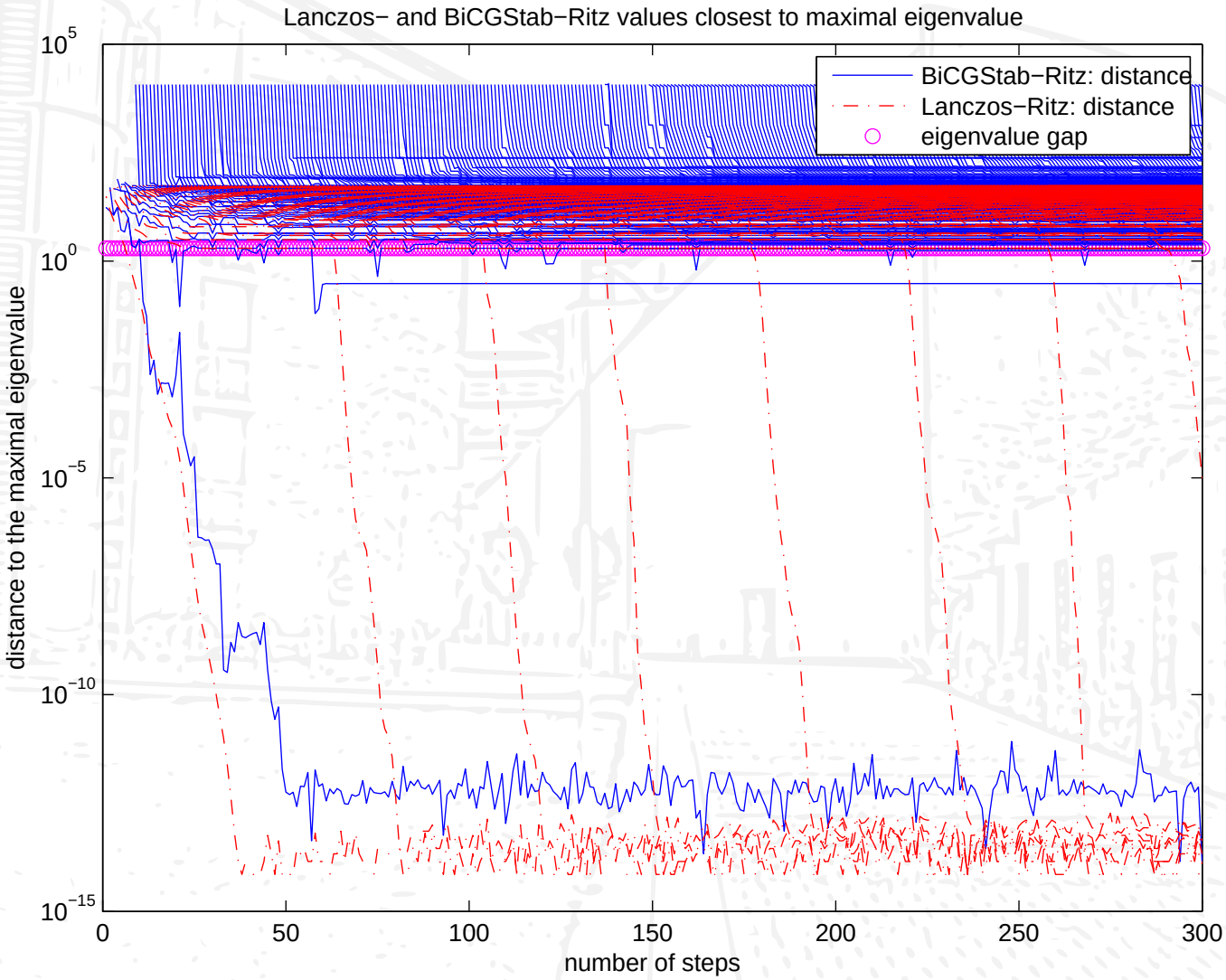

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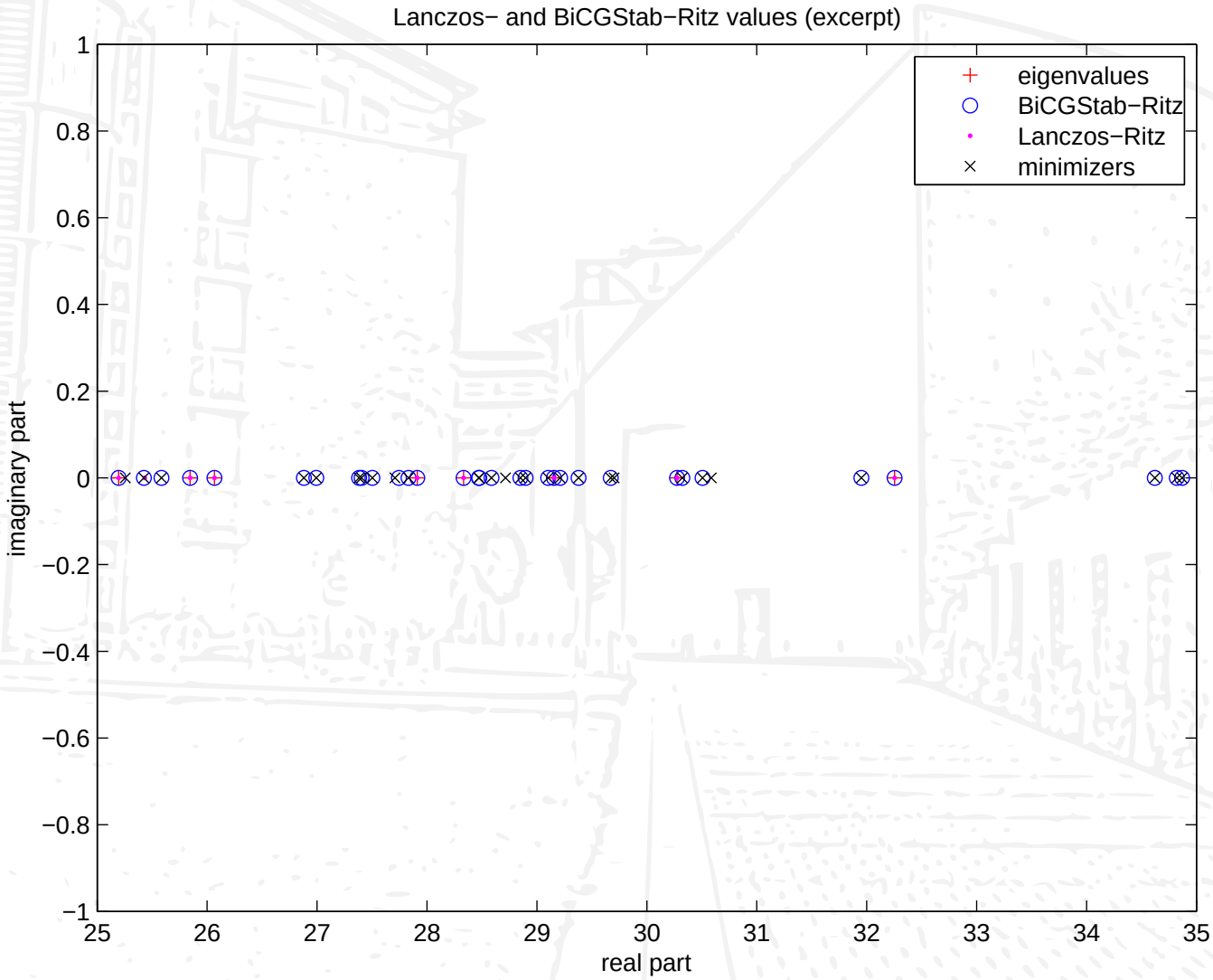

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General setting

We used the stablest variant of IDR(s)Eig to compute the Ritz values of IDR(s)ORes. In all experiments presented, $s = 2$.

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The matrices have been constructed to have a small condition number and such that all eigenvalues are well-conditioned, but these real matrices have imaginary eigenvalues. The matrices are shifted to be positive real or, at least, such that zero is in the left part of the field of values.

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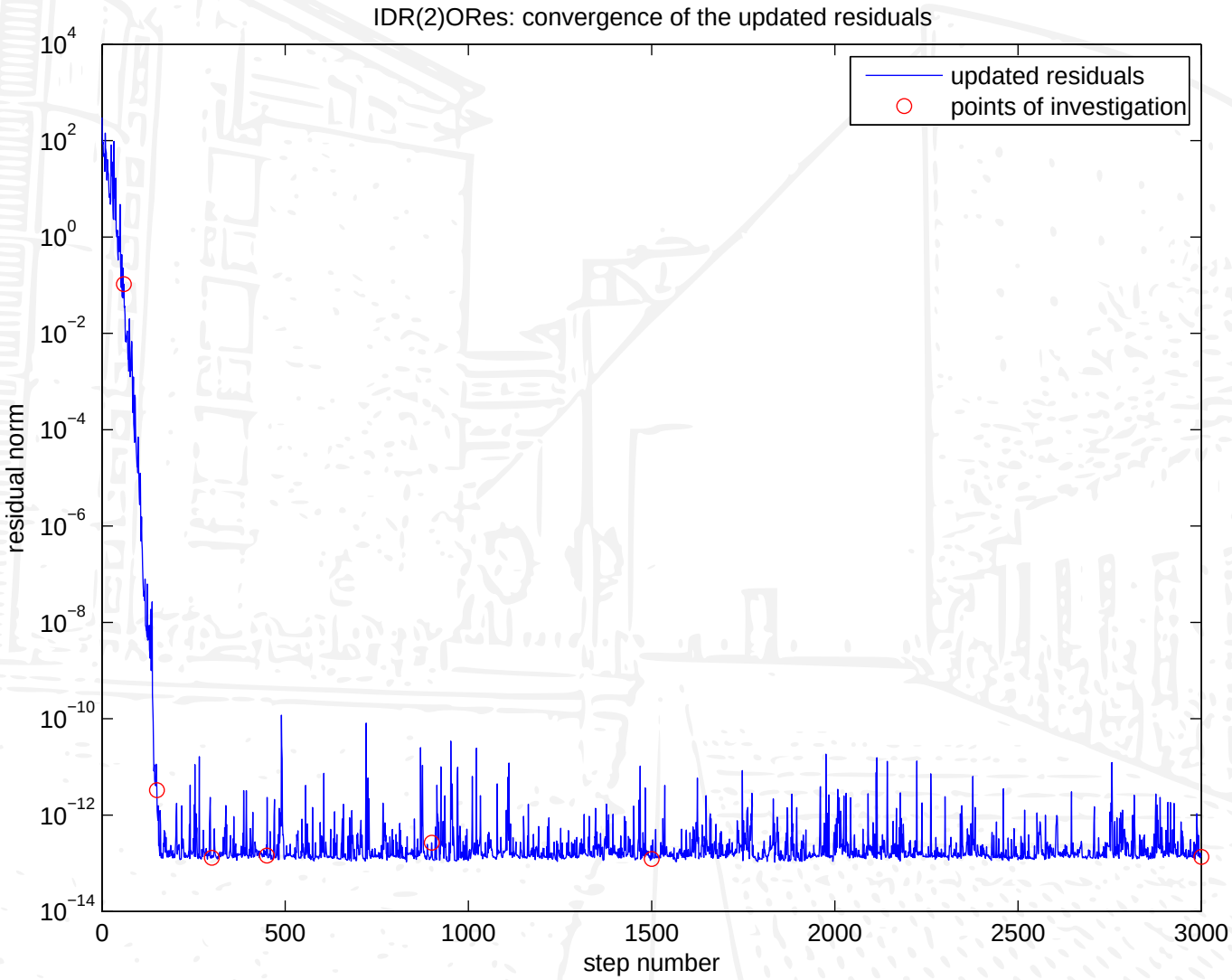
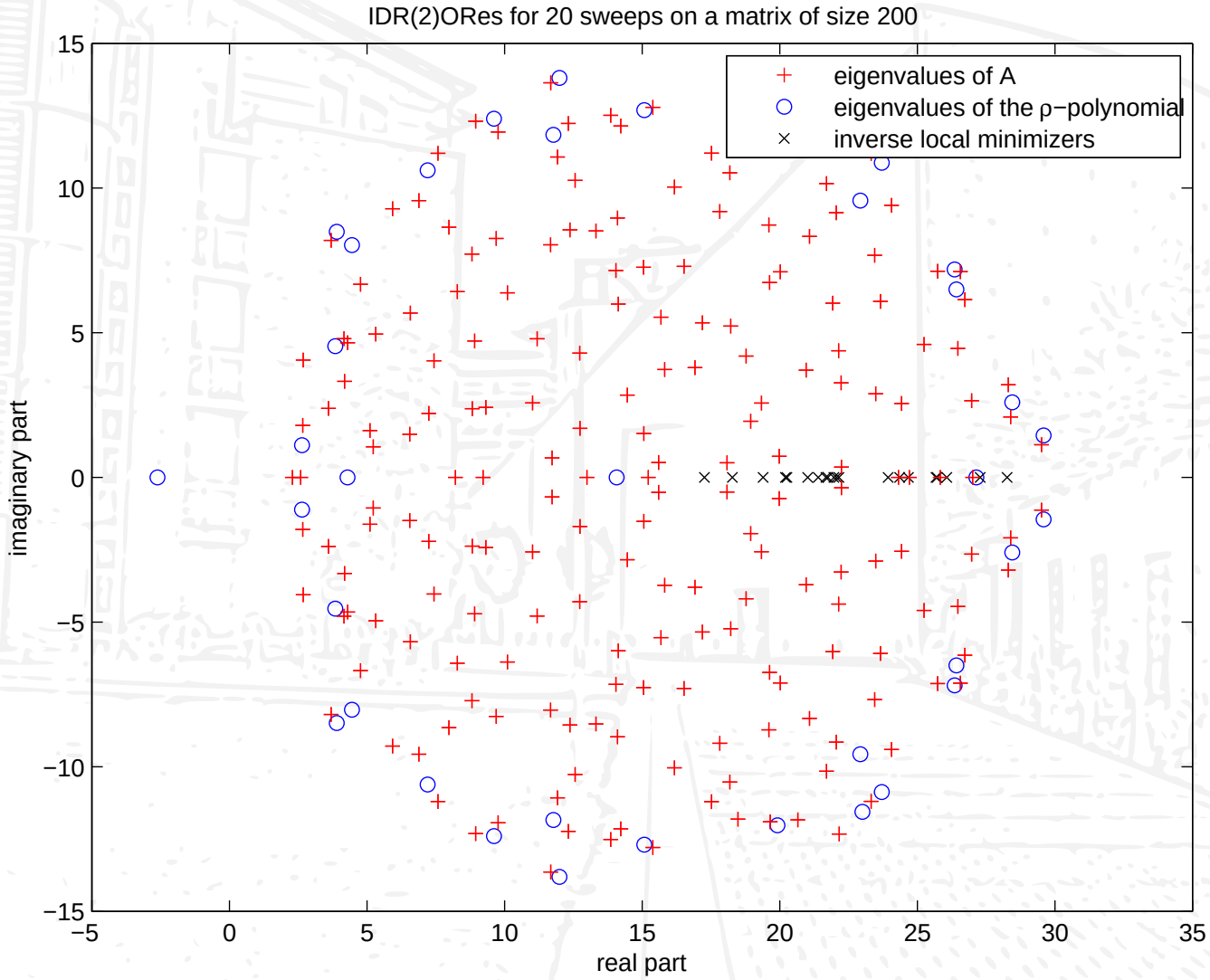
Since for real data all minimizers are real, we have run two sets of experiments with the same matrix and starting residual: The first experiment is based on an orthonormal random real matrix $\mathbf{P} \in \mathbb{R}^{n\times 2}$, the second experiment is based on an orthonormal random complex matrix $\mathbf{P} \in \mathbb{C}^{n\times 2}$.

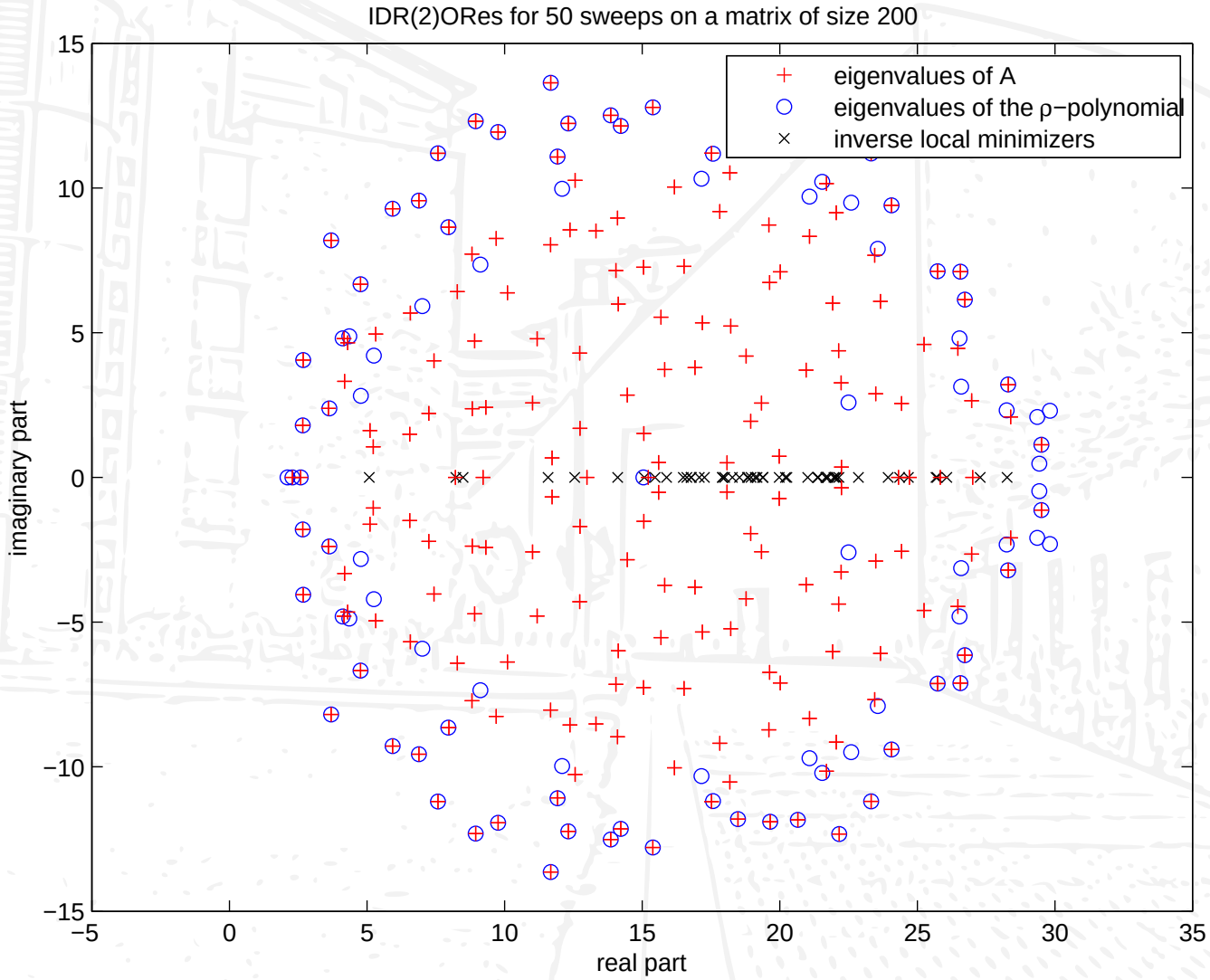
Understanding IDR: Convergence for $s = 2$; $\mathbf{P}$ real



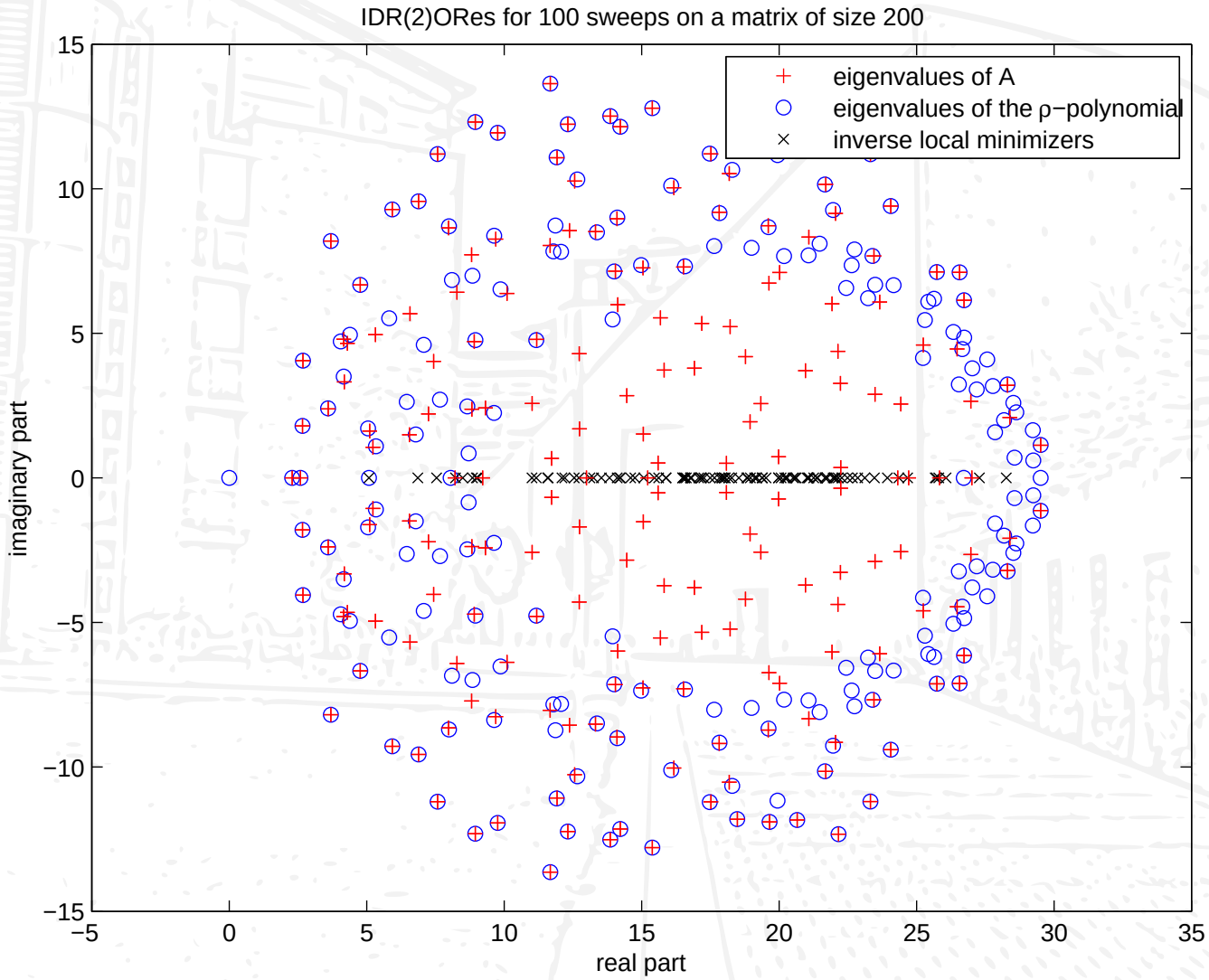
Understanding IDR: 20 steps for $s = 2$; $\mathbf{P}$ real



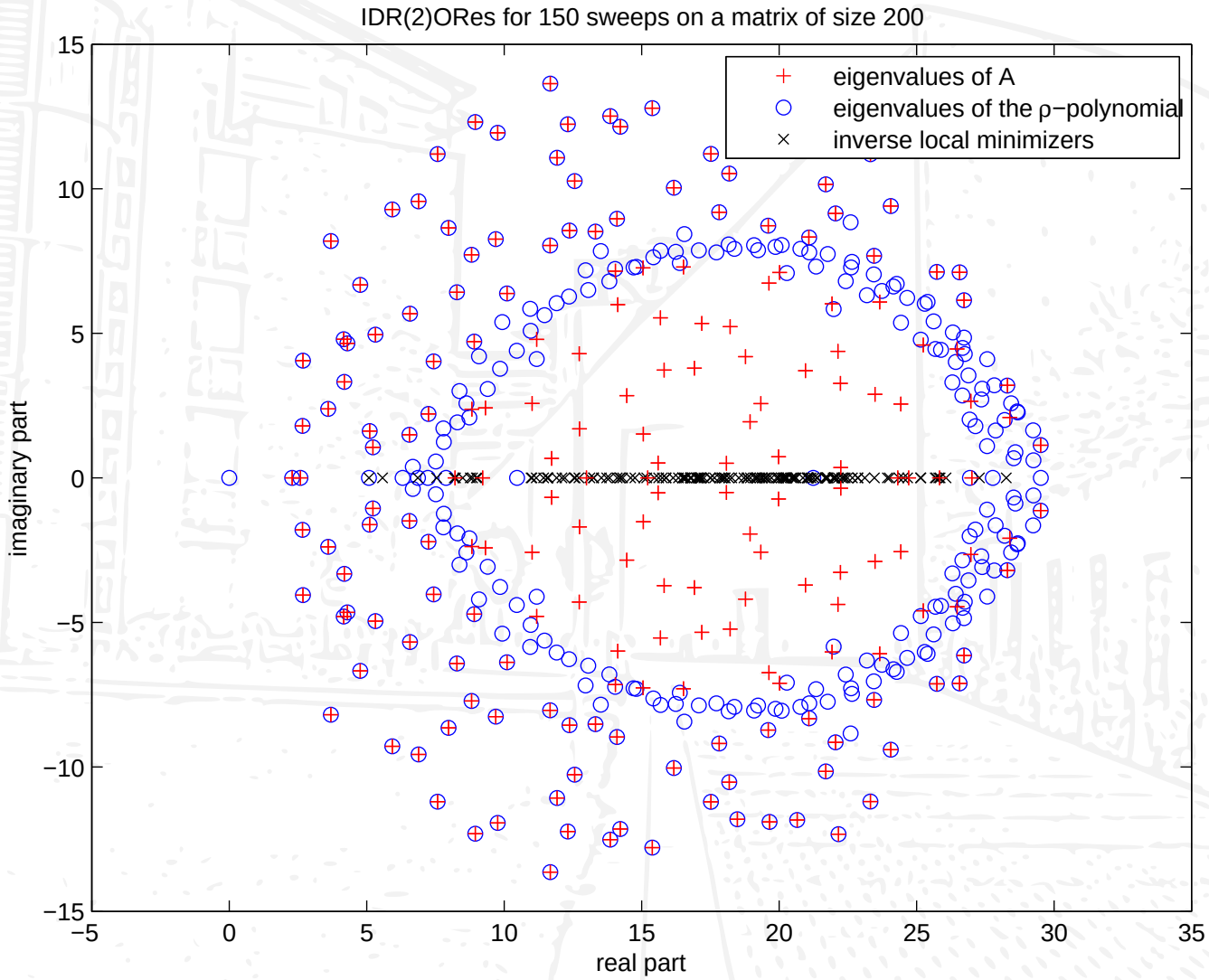
Understanding IDR: 50 steps for $s = 2$; $\mathbf{P}$ real



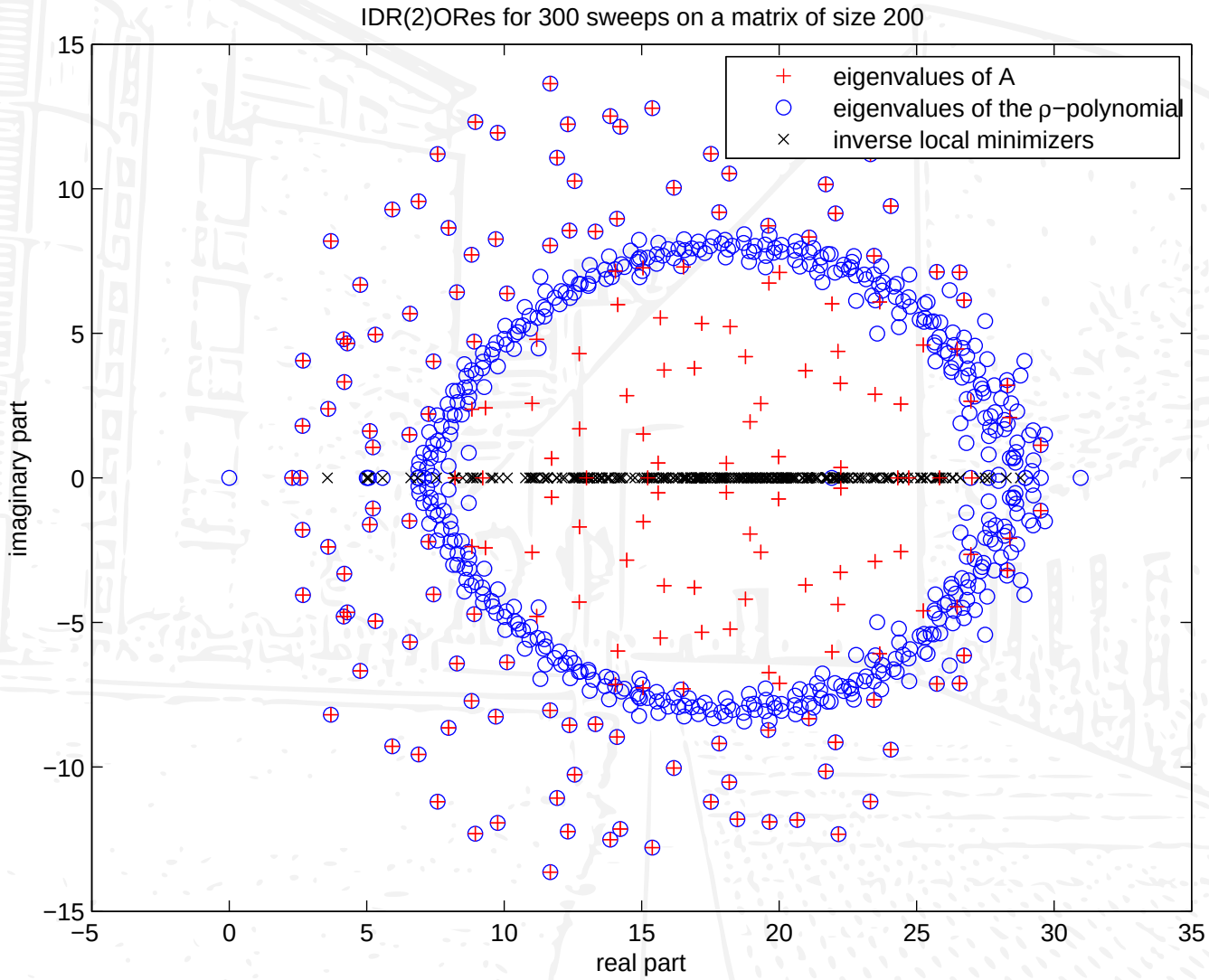
Understanding IDR: 100 steps for $s = 2$; $\mathbf{P}$ real



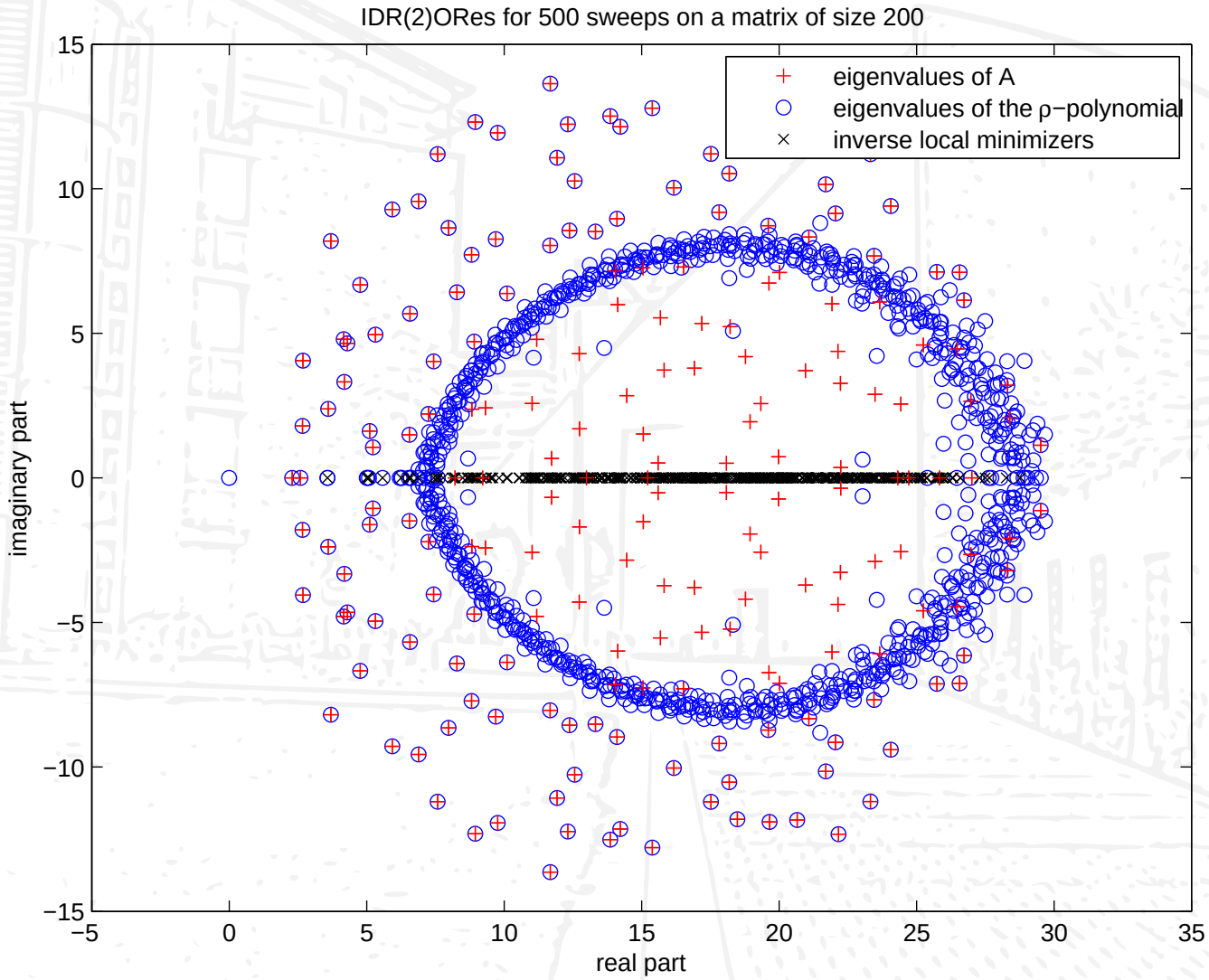
Understanding IDR: 150 steps for $s = 2$; $\mathbf{P}$ real



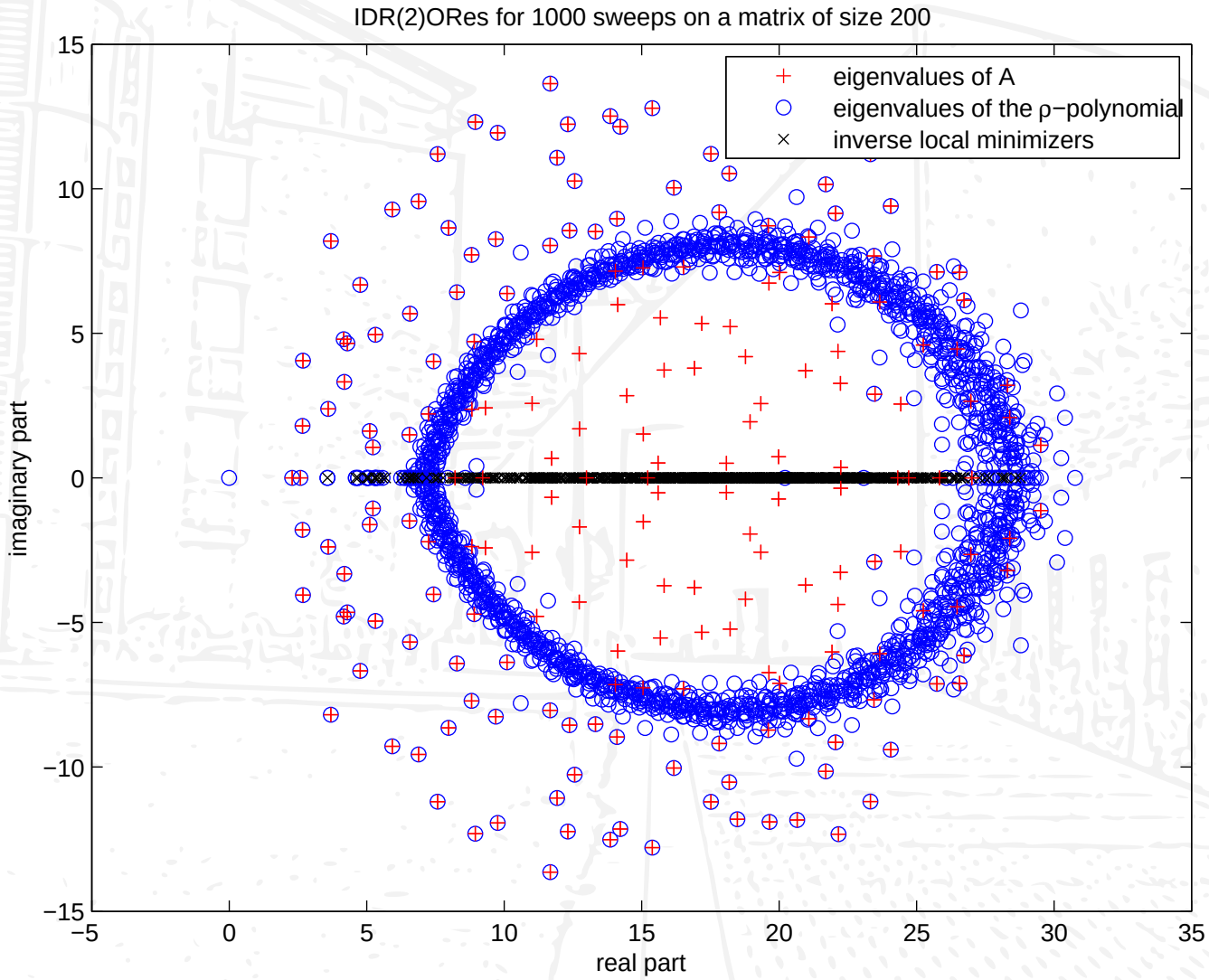
Understanding IDR: 300 steps for $s = 2$; $\mathbf{P}$ real



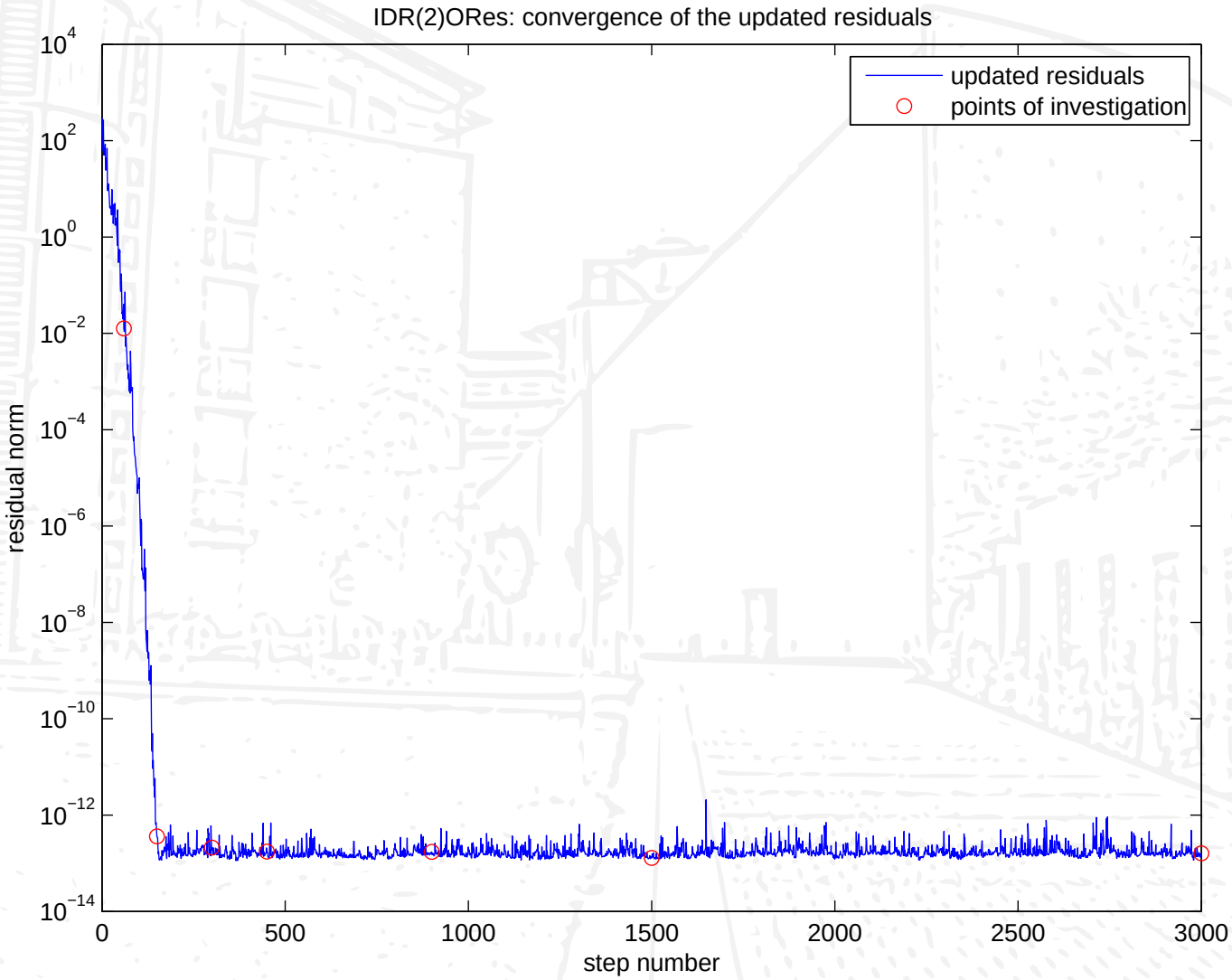
Understanding IDR: 500 steps for $s = 2$; $\mathbf{P}$ real



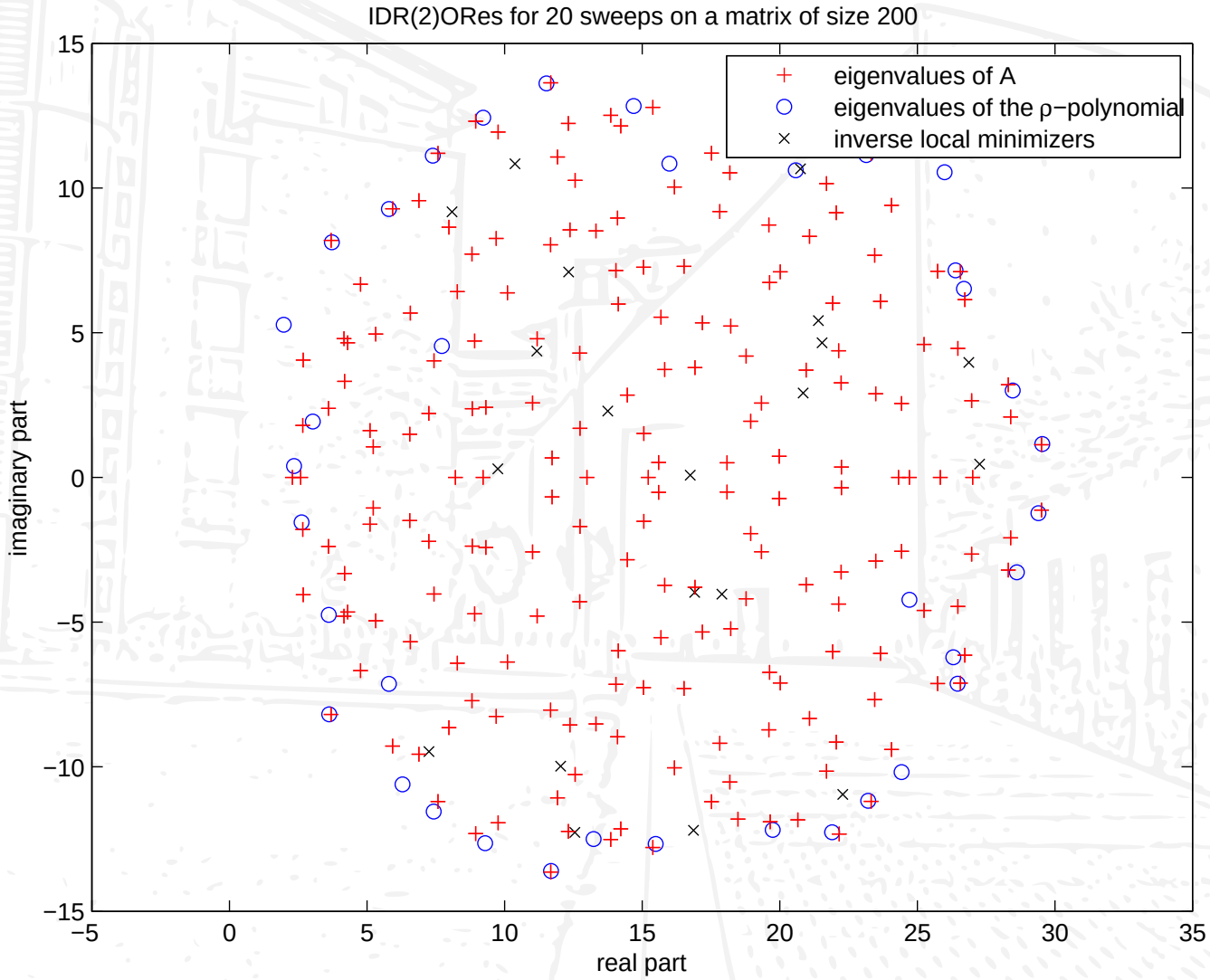
Understanding IDR: 1000 steps for $s = 2$; $\mathbf{P}$ real



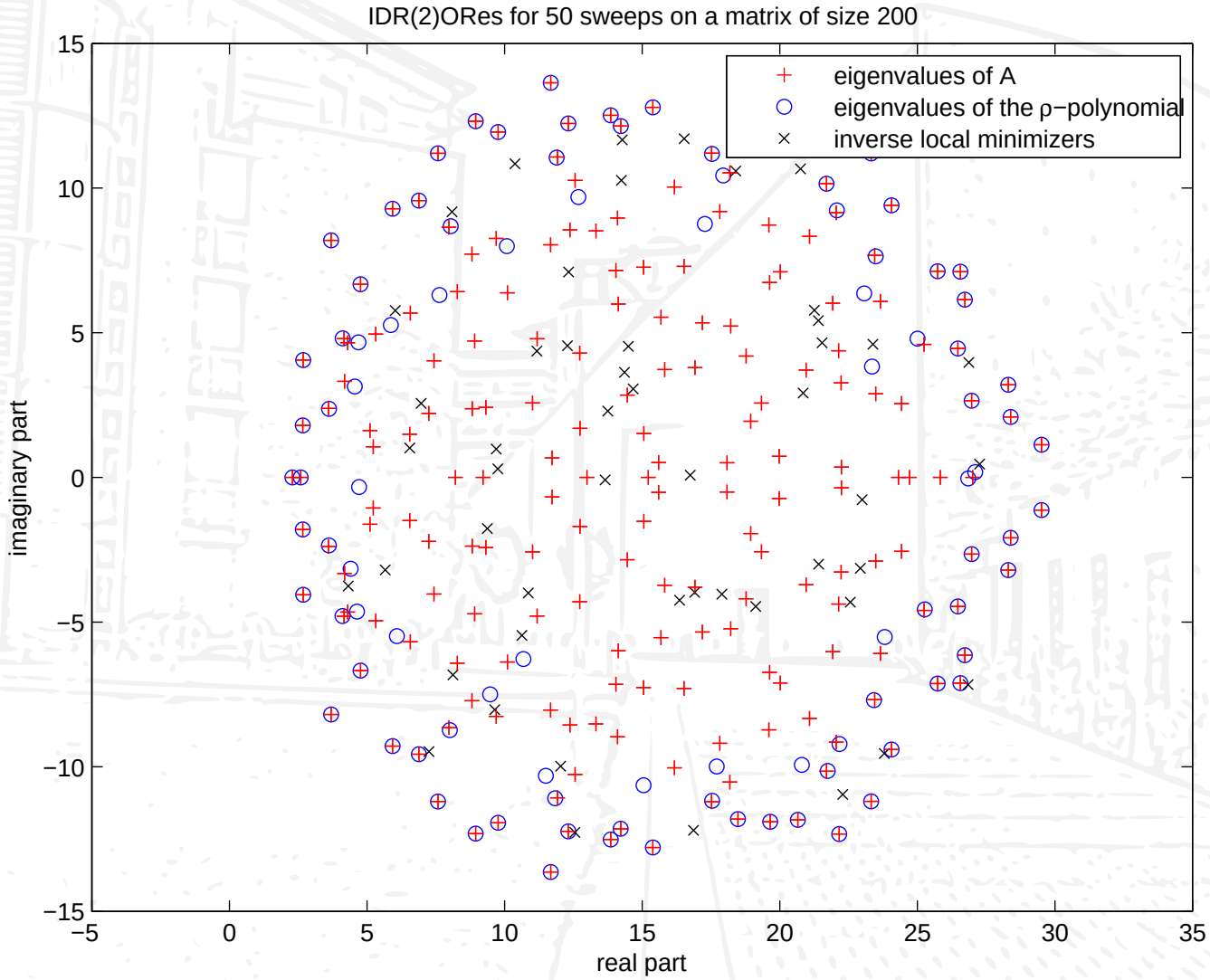
Understanding IDR: Convergence for $s = 2$; $\mathbf{P}$ complex



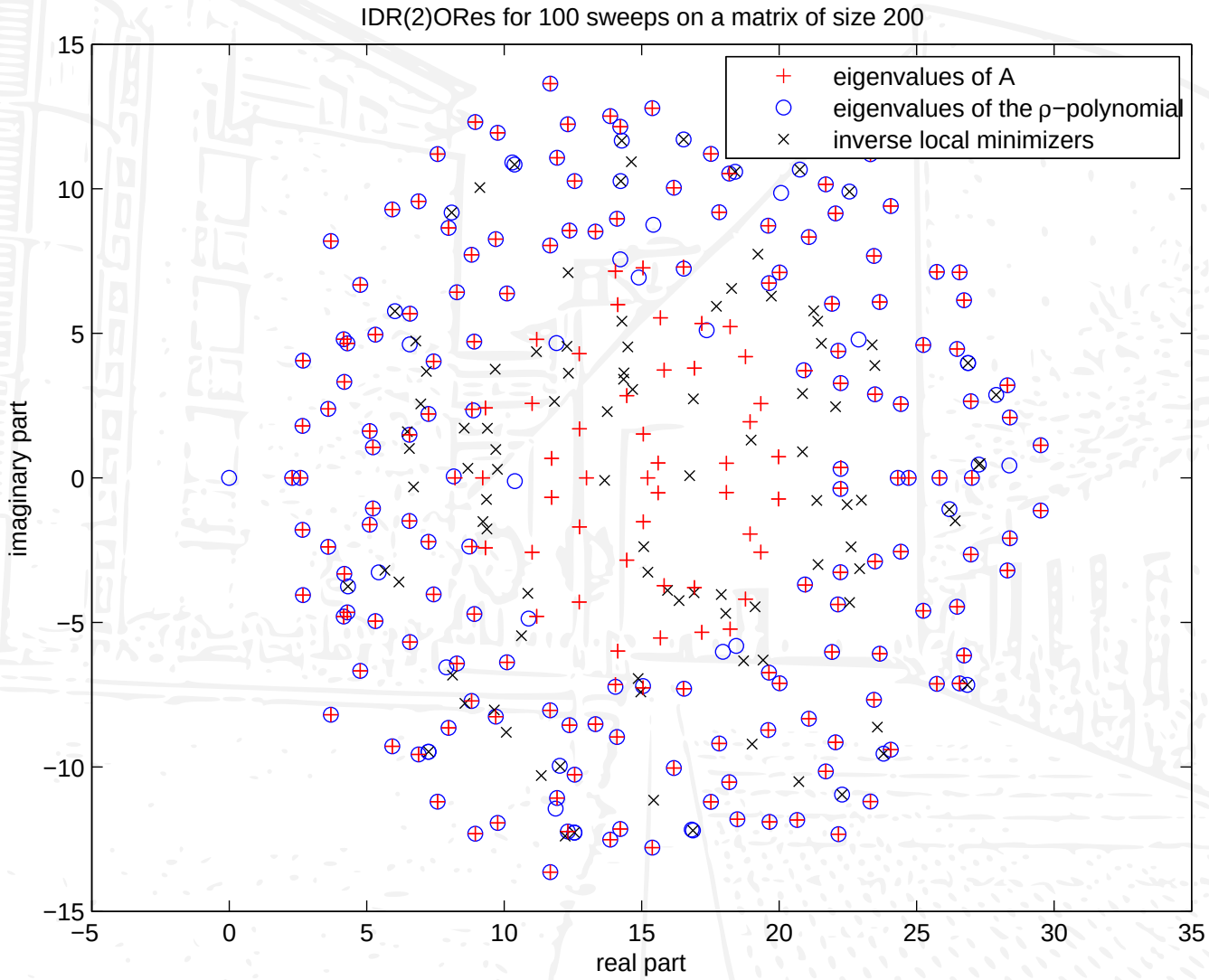
Understanding IDR: 20 steps for $s = 2$; $\mathbf{P}$ complex



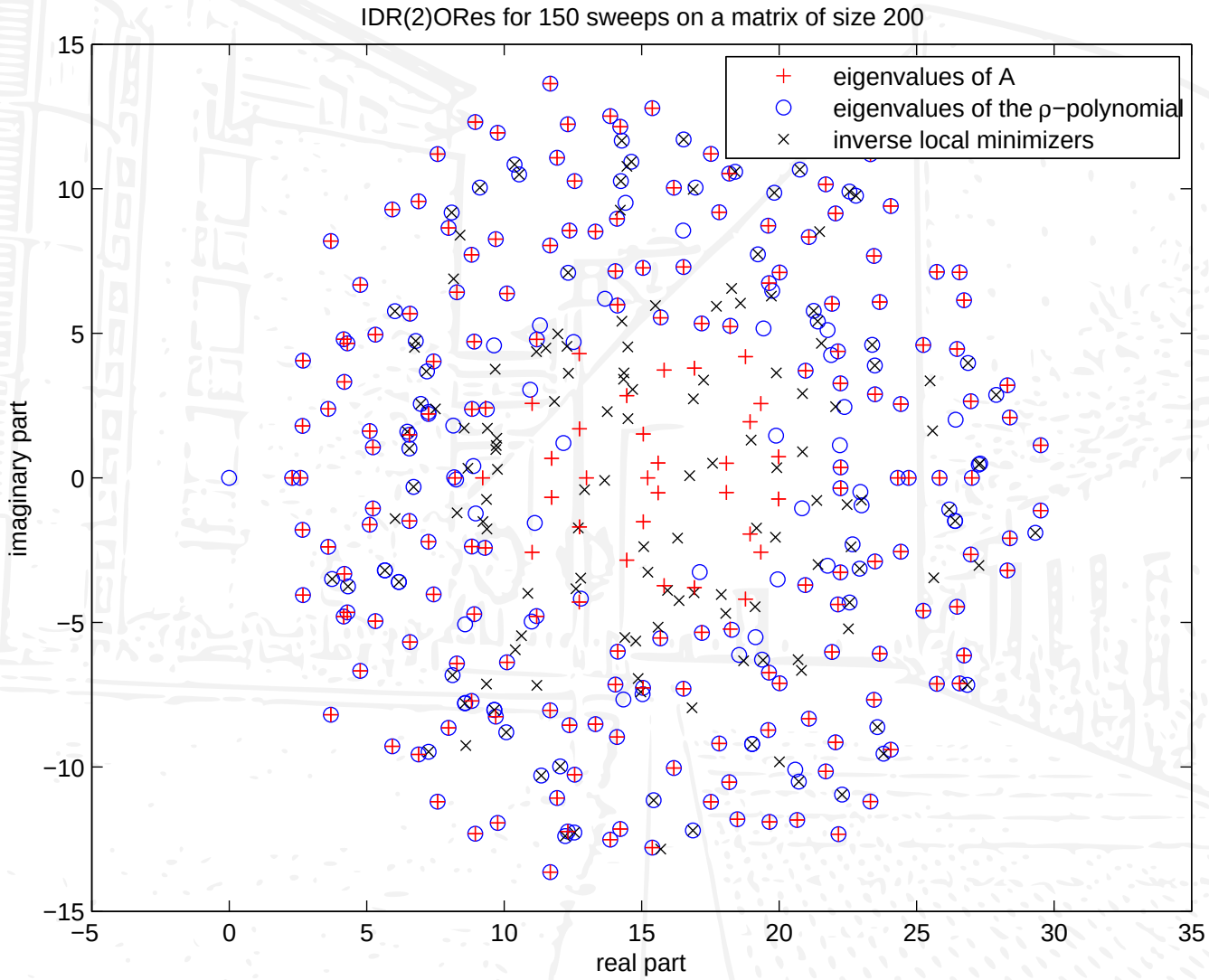
Understanding IDR: 50 steps for $s = 2$; $\mathbf{P}$ complex



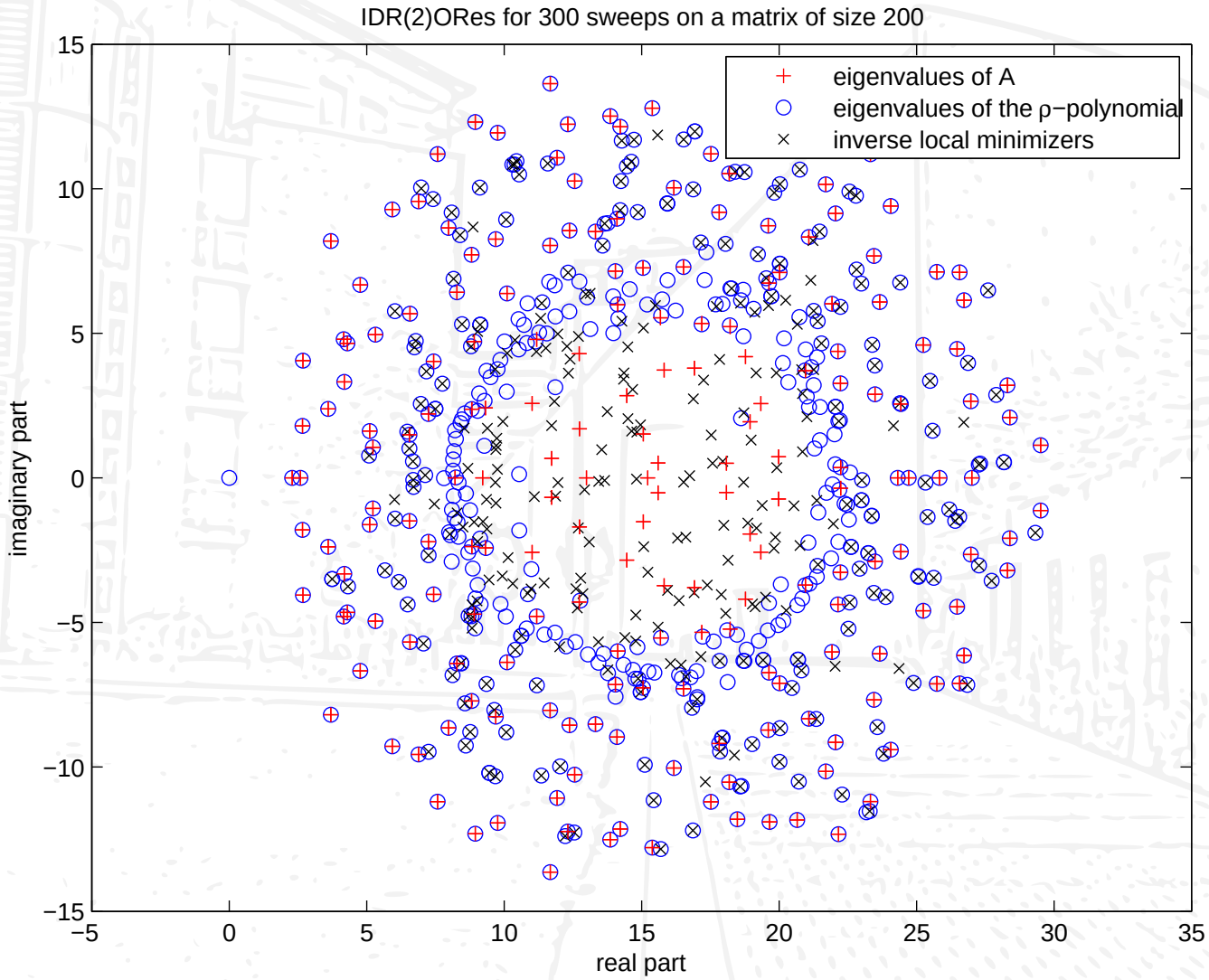
Understanding IDR: 100 steps for $s = 2$; $\mathbf{P}$ complex



Understanding IDR: 150 steps for $s = 2$; $\mathbf{P}$ complex



Understanding IDR: 300 steps for $s = 2$; $\mathbf{P}$ complex



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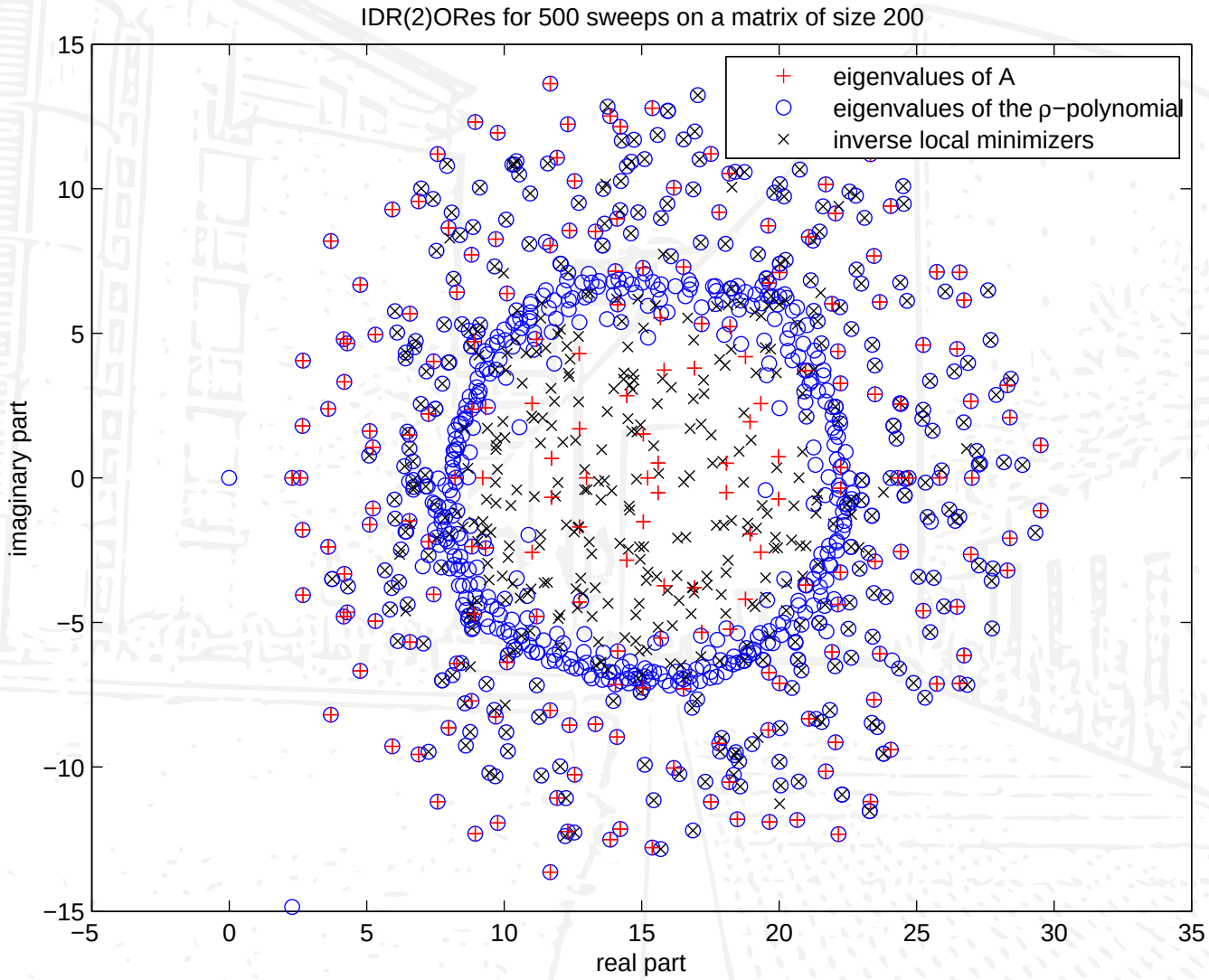

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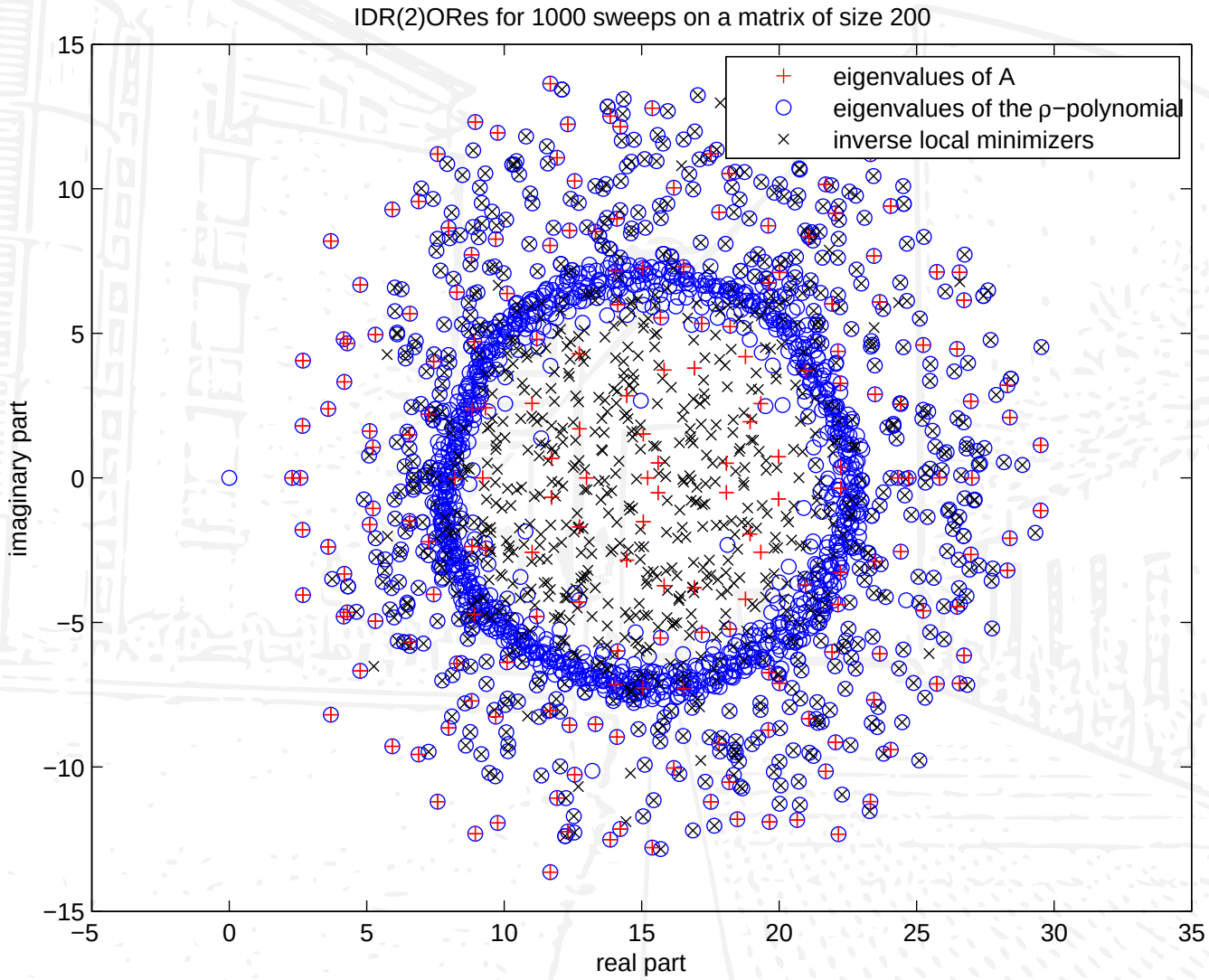

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- We still have to carefully look at the perturbation terms of the underlying perturbed Lanczos process.

Thank you very much for your attention!

どうも有難う御座います

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Bai, Z. (1994).
Error analysis of the Lanczos algorithm for the nonsymmetric eigenvalue problem.
Mathematics of Computation, 62(205):209–226.

Greenbaum, A. (1989).
Behaviour of slightly perturbed Lanczos and conjugate-gradient recurrences.
Linear Algebra and its Applications, 113:7–63.

Greenbaum, A. and Strakoš, Z. (1992).
Predicting the behavior of finite precision Lanczos and conjugate gradient computations.
SIAM J. Matrix Anal. Appl., 13(1):121–137.

Gutknecht, M. H. (1993).
Variants of BICGSTAB for matrices with complex spectrum.
SIAM J. Sci. Comput., 14(5):1020–1033.

Gutknecht, M. H. and Zemke, J.-P. M. (2010).
Eigenvalue computations based on IDR.
Technical Report (to appear 2010).

Meurant, G. (2006).
The Lanczos and Conjugate Gradient Algorithms: From Theory to Finite Precision Computation.
SIAM, Philadelphia.

Meurant, G. and Strakoš, Z. (2006).
The Lanczos and conjugate gradient algorithms in finite precision arithmetic.
Acta Numerica, 15:471–542.

Paige, C. C. (1971).
The Computation of Eigenvalues and Eigenvectors of Very Large Sparse Matrices.
PhD thesis, London University Institute of Computer Science.

Paige, C. C. (1972).
Computational variants of the Lanczos method for the eigenproblem.
J. Inst. Maths Applics, 10:373–381.

Paige, C. C. (1976).
Error analysis of the Lanczos algorithm for tridiagonalizing a symmetric matrix.
J. Inst. Math. Appl., 18(341–349).

Paige, C. C. (1980).
Accuracy and effectiveness of the Lanczos algorithm for the symmetric eigenproblem.
Linear Algebra and its Applications, 34:235–258.

Simoncini, V. and Szyld, D. (2009).
Interpreting IDR as a Petrov-Galerkin method.
Report 09-10-22, Dipartimento di Matematica, Università di Bologna and Department of Mathematics, Temple University, Philadelphia.

Sleijpen, G. L. G. and Abe, K. (2010).
Publication in preparation (January 2010).

Sleijpen, G. L. G., Sonneveld, P., and van Gijzen, M. B. (2008).
Bi-CGSTAB as an induced dimension reduction method.
Reports of the Department of Applied Mathematical Analysis Report 08-07, Delft University of Technology.
ISSN 1389-6520.

Sleijpen, G. L. G. and van Gijzen, M. B. (2009).
Exploiting BiCGstab(ℓ) strategies to induce dimension reduction.
Reports of the Department of Applied Mathematical Analysis Report 09-02, Delft University of Technology.
ISSN 1389-6520.

Sonneveld, P. (2006).
History of IDR: an example of serendipity.
PDF file sent by Peter Sonneveld on Monday, 24th of July 2006.
8 pages; evolved into (Sonneveld, 2008).

Sonneveld, P. (2008).
AGS-IDR-CGS-BiCGSTAB-IDR(s): The circle closed. A case of serendipity.
In *Proceedings of the International Kyoto Forum 2008 on Krylov subspace methods*, pages 1–14.

Sonneveld, P. and van Gijzen, M. B. (2008).
IDR(s): A family of simple and fast algorithms for solving large nonsymmetric systems of linear equations.
SIAM Journal on Scientific Computing, 31(2):1035–1062.
Received Mar. 20, 2007.

Stiefel, E. (1955).
Relaxationsmethoden bester Strategie zur Lösung linearer Gleichungssysteme.
Comment. Math. Helv., 29:157–179.

Tanio, M. and Sugihara, M. (2008).
GIDR(s,l): generalized IDR(s).
In *The 2008 annual conference of the Japan Society for Industrial and Applied Mathematic*, pages 411–412, Chiba, Japan.
(In Japanese).

Tanio, M. and Sugihara, M. (2009).
GBi-CGSTAB(s, L): IDR(s) with higher-order stabilization polynomials.
Technical Report METR 2009-16, Department of Mathematical Informatics, Graduate School of information Science and Technology, University of Tokio.

van Gijzen, M. B. and Sonneveld, P. (2008).
An elegant IDR(s) variant that efficiently exploits bi-orthogonality properties.
Reports of the Department of Applied Mathematical Analysis Report 08-21, Delft University of Technology.
ISSN 1389-6520.

Wesseling, P. and Sonneveld, P. (1980).
Numerical experiments with a multiple grid and a preconditioned Lanczos type method.
In *Approximation Methods for Navier-Stokes Problems*, volume 771 of *Lecture Notes in Mathematics*, pages 543–562. Springer.

Zemke, J.-P. M. (2006).
Hessenberg eigenvalue-eigenmatrix relations.
Linear Algebra and its Applications, 414(2–3):589–606.

Zemke, J.-P. M. (2007).
Abstract perturbed Krylov methods.
Linear Algebra and its Applications, 424(2–3):405–434.

Zhang, S.-L. (1997).
GPBi-CG: generalized product-type methods based on Bi-CG for solving nonsymmetric linear systems.
SIAM Journal on Scientific Computing, 18(2):537–551.