Relations between Rayleigh Quotient Iteration and the Opitz-Larkin Method

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Outline

Rayleigh Quotient Iteration
  John William Strutt’s RQI
  Wielandt’s Inverse Iteration
  “Modern” RQI

The Opitz-Larkin Method
  Classical Root Finding
  Schröder’s and König’s Methods
  The Opitz-Larkin Method

The Hessenberg-Matrix Point Of View
  …and what about Jenkins-Traub?
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In the second edition of the first volume of his book “The Theory of Sound” (Strutt, 1894), John William Strutt, 3rd Baron Rayleigh, included on page 110 the following passage:
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The stationary property of the roots of Lagrange’s determinant (3) § 84, suggests a general method of approximating to their values. Beginning with assumed rough approximations to the ratios \( A_1 : A_2 : A_3 \) \ldots we may calculate a first approximation to \( p^2 \) from

\[
p^2 = \frac{1}{2} c_{11} A_1^2 + \frac{1}{2} c_{22} A_2^2 + \ldots + c_{12} A_1 A_2 + \ldots
\]

\[
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\]

With this value of \( p^2 \) we may recalculate the ratios \( A_1 : A_2 \ldots \) from any \((m - 1)\) of equations (5) § 84, then again by application of (3) determine an improved value of \( p^2 \), and so on.]
In modern notation, Lord Rayleigh starts with an approximate eigenvector $v_k$, $k = 0$, of a Hermitean matrix (Hermitean pencil), computes its Rayleigh quotient

$$\rho(v_k) := \frac{v_k^H Av_k}{v_k^H v_k},$$

and iterates for some suitably chosen $j \in \{1, 2, \ldots, n\}$,

$$v_{k+1} = (A - \rho(v_k)I_n)^{-1}e_j \| (A - \rho(v_k)I_n)^{-1}e_j \|,$$

where $j$ may vary, depending on the computed approximate eigenvector.

The Rayleigh quotient uniquely solves the least squares problem

$$\rho(v_k) = \arg\min_{\rho \in \mathbb{C}} \| Av_k - v_k \rho \|.$$
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The shift can be updated by using the approximate eigenvalues obtained by the shift update strategy

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The latter variant is described in (Wielandt, 1944, Seite 9, Formel (20)) and converges locally quadratically.
Modern variants of RQI

Combination gives (symmetric/Hermitean) RQI:

\[ v_{k+1} = \frac{(A - \rho(v_k)I_n)^{-1}v_k}{\| (A - \rho(v_k)I_n)^{-1}v_k \|}, \quad k = 0, 1, \ldots \]

This iteration is also used for nonsymmetric A.
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Crandall was the first who investigated the three variants (the original Rayleigh quotient iteration; inverse iteration with fixed shift; symmetric RQI), see (Crandall, 1951).
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Ostrowski proved that unsymmetric RQI still has a quadratic convergence rate, (Ostrowski, 1959e). In (Ostrowski, 1959c), he devised two-sided RQI:

\[ \rho(\mathbf{w}_k, \mathbf{v}_k) := \frac{\mathbf{w}_k^H \mathbf{A} \mathbf{v}_k}{\mathbf{w}_k^H \mathbf{v}_k}, \quad \mathbf{v}_{k+1} = (\mathbf{A} - \rho(\mathbf{w}_k, \mathbf{v}_k)\mathbf{I}_n)^{-1}\mathbf{v}_k, \quad k = 0, 1, \ldots \]

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This trick recovers the cubic convergence rate of RQI at the expense of an additional system.
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Classical methods

Methods for the **computation of a root** of a rational function

\[ f : \mathbb{C} \to \mathbb{C}, \quad f(z) := \frac{p(z)}{q(z)}, \quad p, q \in \mathbb{P}_m \]

include Newton’s method

\[ z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)} \]

and the secant method:

\[ z_{k+1} = z_k - \frac{f(z_k)}{[z_k, z_{k-1}]f}. \]
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The secant method has R-order of convergence given by the golden ratio

\[ \phi := \frac{1 + \sqrt{5}}{2} \approx 1.618. \]
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Two steps of the secant method are as costly as one step of Newton’s method. This makes the secant method the winner:

\[ \phi^2 = \phi + 1 \approx 2.618 > 2. \]
Schröder’s and König’s methods

Newton’s method has been generalized to incorporate higher order derivatives and to exhibit a higher order of convergence. Well-known generalized Newton’s methods are Halley’s and Laguerre’s methods.
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This family is nowadays known as “König’s method”:

$$z_{k+1} = z_k + s \frac{(1/f)^{(s-1)}(z_k)}{(1/f)^{(s)}(z_k)}, \quad s = 1, 2, \ldots$$
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$$

König’s method for $s = 1$ is Newton’s method,

$$
    z_{k+1} = z_k + \frac{(1/f)(z_k)}{(1/f)'(z_k)} = z_k - \frac{1/f(z_k)}{f'(z_k)/(f(z_k))^2} = z_k - \frac{f(z_k)}{f'(z_k)}.
$$
There is a natural extension of König’s method using \textit{divided differences} in place of the \textit{derivatives}. 
The Opitz-Larkin method

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He published few additional papers on the subject (including his most famous “Steigungsmatrizen” paper). A more complete presentation can be found in his “Habilitationsschrift”. There, he even pointed out the connection to König’s method.

Independently, 23 years later F. M. Larkin re-developed Opitz’ method, see (Larkin, 1981) and the predecessor (Larkin, 1980).

We will refer to this method as the Opitz-Larkin method. The Opitz-Larkin method is based on iterations of the form

$$x_{k+1} = z_k + \left[ z_1, z_2, ..., z_{k-1} \right] \left( \frac{1}{f} \right) \left[ z_1, z_2, ..., z_{k-1}, z_k \right] \left( \frac{1}{f} \right).$$
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\[ x_{k+1} = z_k + \frac{[z_1, z_2, \ldots, z_{k-1}, z_k]}{[z_1, z_2, \ldots, z_{k-1}, z_k]} \frac{1}{f}. \]
Mostly, the $z_i$ are all distinct and the next iterate is used as new evaluation point

$$z_{k+1} = x_{k+1},$$

$$z_{k+1} = z_k + \frac{[z_1, z_2, \ldots, z_{k-1}](1/f)}{[z_1, z_2, \ldots, z_{k-1}, z_k](1/f)}.$$
Mostly, the $z_i$ are all distinct and the next iterate is used as new evaluation point $z_{k+1} = x_{k+1}$,

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This variant of the Opitz-Larkin method converges with $R$-order 2.
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This variant of the Opitz-Larkin method converges with R-order 2.

Frequently, the Opitz-Larkin method is used with truncation:

$$z_{k+1} = z_k + \frac{[z_{k-p}, \ldots, z_{k-1}](1/f)}{[z_{k-p}, \ldots, z_{k-1}, z_k](1/f)},$$

see (Opitz, 1958, Seite 277, Gleichung (9)) and (Larkin, 1981, Section 4, pages 98–99).
The Opitz-Larkin method

It is possible to use confluent divided differences, i.e., multiple points of evaluation, i.e., higher order derivatives of $1/f$. 

$$
\frac{1}{2} \sum_{j=1}^{s-1} \left( \sum_{i=0}^{j} \binom{s-1}{i} \frac{1}{f(j)} \right) = 
\frac{1}{s} \sum_{j=1}^{s-1} \left( \sum_{i=0}^{j} \binom{s-1}{i} \frac{1}{f(j)} \right) 
$$
It is possible to use confluent divided differences, i.e., multiple points of evaluation, i.e., higher order derivatives of $1/f$.

When we use only confluent divided differences in the truncated Opitz-Larkin method with truncation parameter $p = s$, we recover König’s method:

$$z_{k+1} = z_k + \sum_{i=0}^{s} \left[ \frac{z_k, \ldots, z_k}{1/f} \right] = z_k + \frac{(1/f)^{(s-1)}(z_k)/(s-1)!}{(1/f)^{(s)}(z_k)}/s! = z_k + s \frac{(1/f)^{(s-1)}(z_k)}{(1/f)^{(s)}(z_k)}.$$
Truncated Opitz-Larkin with $p = 1$ is the secant method,

\[
z_{k+1} = z_k + \frac{[z_{k-1}] (1/f)}{[z_{k-1}, z_k] (1/f)}
\]

\[= z_k + \frac{1}{f(z_{k-1})} \cdot \frac{z_{k-1} - z_k}{1/f(z_{k-1}) - 1/f(z_k)}
\]

\[= z_k + \frac{f(z_k)f(z_{k-1})}{f(z_{k-1})} \cdot \frac{z_{k-1} - z_k}{f(z_k) - f(z_{k-1})}
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Confluent truncated Opitz-Larkin with $p = 1$ is Newton’s method.
The Opitz-Larkin method

In general, the Opitz-Larkin method is closely connected to rational interpolation of the inverse function (Larkin, 1981, Theorem 1, page 96):
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**Theorem (Larkin 1981)**

*If, for any integer \( k > 1 \), there exists a rational function of the form*

\[
 r_k(z) = \frac{q_d(z)}{z - \alpha}, \quad \forall \ z,
\]

*where \( q_d \) is a polynomial of degree \( d \leq k - 2 \), such that \( q_d(\alpha) \neq 0 \) and*

\[
 r_k(z_j) = f(z_j)^{-1}, \quad j = 1, 2, \ldots, k,
\]

*then*

\[
 z_k + \frac{[z_1, z_2, \ldots, z_{k-1}](1/f)}{[z_1, z_2, \ldots, z_{k-1}, z_k](1/f)} = \alpha.
\]
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The Hessenberg-Matrix Point Of View
  .... and what about Jenkins-Traub?
By the **implicit Q-Theorem** we obtain a **unique** Hessenberg matrix given nonderogatory \( A \in \mathbb{C}^{n \times n} \) and \( q \in \mathbb{C}^n \) if we fix the **signs** of the elements in the lower diagonal, e.g., to be non-negative real.
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We use the implicit Q-Theorem to **unitarily transform** the pair $(A, q)$ with $\|q\|_2 = 1$ to the pair $(H_n, e_1)$, where $H_n$ is **upper Hessenberg** and $e_1$ denotes the first standard unit vector.
The Hessenberg-Matrix Point Of View

Simplification

By the implicit Q-Theorem we obtain a unique Hessenberg matrix given nonderogatory \( A \in \mathbb{C}^{n \times n} \) and \( q \in \mathbb{C}^n \) if we fix the signs of the elements in the lower diagonal, e.g., to be non-negative real.

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The following Matlab-code gives the transformed pair:

\[
\begin{align*}
[Q,R] & = \text{qr}(q); \\
[P,H] & = \text{hess}(Q' \ast A \ast Q);
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[Q,R] = qr(q);
[P,H] = hess(Q'*A*Q);
signs = sign(diag(H,-1));
S = diag(cumprod([1;signs]));
H = S'*H*S;
```

Any left vector is modified accordingly.
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\]

Any left vector is modified accordingly.
We set $zH_n := (zI_n - H_n)$. By the first resolvent identity (Chatelin, 1993)

\[(z_1 H_n)^{-1} (z_2 H_n)^{-1} = (z_1 I_n - H_n)^{-1} (z_2 I_n - H_n)^{-1} \]

\[= \frac{(z_1 H_n)^{-1} - (z_2 H_n)^{-1}}{z_2 - z_1} = -[z_1, z_2](zH_n)^{-1}. \]
Simplification

We set \( z^n := (zI_n - H_n) \). By the first resolvent identity (Chatelin, 1993)

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= \frac{(z_1 H_n)^{-1} - (z_2 H_n)^{-1}}{z_2 - z_1} = -[z_1, z_2] (z H_n)^{-1}.
\] (1b)

The first resolvent identity is based on the trivial observation that

\[
(z_2 I_n - H_n) - (z_1 I_n - H_n) = (z_2 - z_1) I_n.
\]
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We set \( z^H_n := (zI_n - H_n) \). By the first resolvent identity (Chatelin, 1993)

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\]

Generalization (see also (Dekker and Traub, 1971)):

\[
\prod_{i=1}^{k} (z_i H_n)^{-1} = (-1)^{k-1} [z_1, \ldots, z_k] (z H_n)^{-1}.
\]

(2)
Simplification

We set $z H_n := (z I_n - H_n)$. By the first resolvent identity (Chatelin, 1993)

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(z_1 H_n)^{-1} (z_2 H_n)^{-1} = (z_1 I_n - H_n)^{-1} (z_2 I_n - H_n)^{-1}
$$

$$
= \left( z_1 H_n \right)^{-1} - \left( z_2 H_n \right)^{-1} = \frac{z_1 H_n}{z_2 - z_1} = -[z_1, z_2] (z H_n)^{-1}.
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Confluent divided differences are well-defined.
Simplification

For simplicity we assume that $H_n$ is unreduced.
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$$h_{i:j} := \prod_{\ell = i}^{j} h_{\ell + 1, \ell}.$$
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$$h_{i:j} := \prod_{\ell = i}^{j} h_{\ell+1, \ell}.$$ 

Polynomial vectors $\nu$ and $\tilde{\nu}$ are defined by

$$\nu(z) := \left( \frac{\chi_{j+1:n}(z)}{h_{j:n-1}} \right)^n_{j=1} \quad \text{and} \quad \tilde{\nu}(z) := \left( \frac{\chi_{1:j-1}(z)}{h_{1:j-1}} \right)^n_{j=1}. \quad (3)$$
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The elements are $\nu_j(z)$ and $\tilde{\nu}_j(z), j = 1, \ldots, n$. Observe that $\nu_n \equiv 1 \equiv \tilde{\nu}_1$. 
The Hessenberg-Matrix Point Of View

Simplification

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The elements are $\nu_j(z)$ and $\tilde{\nu}_j(z)$, $j = 1, \ldots, n$. Observe that $\nu_n \equiv 1 \equiv \tilde{\nu}_1$.

The polynomials $\chi_{i:j}$ are the characteristic polynomials of submatrices of $H_n$,

$$\chi_{i:j}(z) := \det \left( zH_{i:j} \right) = \det \left( zI_{j-i+1} - H_{i:j} \right).$$
For $z$ in the resolvent set

$$\left( \frac{z}{H_n} \right) \nu(z) = \frac{\chi(z)}{h_{1:n-1}} e_1 \iff \nu(z) h_{1:n-1} = \frac{\chi(z)}{h_{1:n-1}} \left( \frac{z}{H_n} \right) e_1,$$

(4a)

$$\tilde{\nu}(z)^T \left( \frac{z}{H_n} \right) = e_n^T \frac{\chi(z)}{h_{1:n-1}} \iff h_{1:n-1} \tilde{\nu}(z)^T = e_n^T \left( \frac{z}{H_n} \right)^{-1}.$$

(4b)
For $z$ in the resolvent set

$$\left(zH_n\right)\nu(z) = \frac{\chi(z)}{h_{1:n-1}}e_1 \quad \Leftrightarrow \quad \nu(z)h_{1:n-1} = \frac{\chi(z)}{h_{1:n-1}}e_1 = \left(zH_n\right)^{-1}e_1, \quad (4a)$$

$$\tilde{\nu}(z)^T(zH_n) = e_n^T \frac{\chi(z)}{h_{1:n-1}} \quad \Leftrightarrow \quad h_{1:n-1}\tilde{\nu}(z)^T = e_n^T(zH_n)^{-1} \quad (4b)$$

The repeated application of resolvents to $e_1$ results in

$$\left(\prod_{i=1}^{k} (z_iH_n)^{-1}\right)e_1 = (-1)^{k-1}[z_1, \ldots, z_k](zH_n)^{-1}e_1 \quad (5)$$

$$= (-1)^{k-1}[z_1, \ldots, z_k] \frac{\nu(z)h_{1:n-1}}{\chi(z)}. \quad (6)$$
Simplification

For $z$ in the resolvent set

$$
(zH_n)\nu(z) = \frac{\chi(z)}{h_{1:n-1}} e_1 \quad \Leftrightarrow \quad \frac{\nu(z)h_{1:n-1}}{\chi(z)} = (zH_n)^{-1} e_1, \quad (4a)
$$

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\nu(z)^T (zH_n) = e_n^T \frac{\chi(z)}{h_{1:n-1}} \quad \Leftrightarrow \quad \frac{h_{1:n-1}\nu(z)^T}{\chi(z)} = e_n^T (zH_n)^{-1}. \quad (4b)
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= (-1)^{k-1}[z_1, \ldots, z_k] \frac{\nu(z)h_{1:n-1}}{\chi(z)}. \quad (6)
$$

Note that $zI_n = zH_n = zI_n - (zI_n - H_n) = H_n$, i.e., $H_n(zH_n)^{-1} = z(zH_n)^{-1} - I_n$. 
For the sake of eased understanding, we look at inverse iteration with a two-sided Rayleigh quotient where the left vector is the last standard unit vector $e_n^T$. 

\begin{align*}
  v_{k+1} &= \left( \prod_{i=1}^{k} (z_i H_n) \right)^{-1} e_1, \\
  x_{k+1} &= e_n^T (\prod_{i=1}^{k} (z_i H_n) ) e_1,
\end{align*}

and thus the approximate eigenvalues are given by the Opitz-Larkin method:

\begin{align*}
  x_{k+1} &= e_n^T (\prod_{i=1}^{k} (z_i H_n) ) e_1 = e_n^T (z_k I_n - (z_k H_n)) (\prod_{i=1}^{k} (z_i H_n) ) e_1.
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For the sake of eased understanding, we look at inverse iteration with a two-sided Rayleigh quotient where the left vector is the last standard unit vector $e_n^T$. For this method we have the iterates

$$v_{k+1} = \left( \prod_{i=1}^{k} (z_i H_n)^{-1} \right) e_1, \quad x_{k+1} = \frac{e_n^T H_n v_{k+1}}{e_n^T v_{k+1}}.$$
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and thus the approximate eigenvalues are given by the Opitz-Larkin method:

$$x_{k+1} = \frac{e_n^T H_n \left( \prod_{i=1}^{k} (z_i H_n)^{-1} \right) e_1}{e_n^T \left( \prod_{i=1}^{k} (z_i H_n)^{-1} \right) e_1} = \frac{e_n^T (z_k I_n - (z_k H_n)) \left( \prod_{i=1}^{k} (z_i H_n)^{-1} \right) e_1}{e_n^T \left( \prod_{i=1}^{k} (z_i H_n)^{-1} \right) e_1}$$

$$= z_k - \frac{e_n^T z_k H_n \left( \prod_{i=1}^{k} (z_i H_n)^{-1} \right) e_1}{e_n^T \left( \prod_{i=1}^{k} (z_i H_n)^{-1} \right) e_1}$$

$$= z_k + \frac{[z_1, \ldots, z_{k-1}, z_k]}{[z_1, \ldots, z_{k-1}, z_k]} (1/\chi).$$

(7a)
Simplification

When we update the shifts by choosing $z_{k+1} = x_{k+1}$ we obtain the standard variant of the Opitz-Larkin method. This method has asymptotically second order convergence against the roots of the characteristic polynomial $\chi$. 
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Inverse iteration with fixed shift $\tau = z_1 = z_2 = \ldots = z_k$ results in the recurrence

$$x_{k+1} = \tau + \frac{[\tau, \ldots, \tau](1/\chi)}{[\tau, \ldots, \tau, \tau](1/\chi)} = \tau + k \frac{(1/\chi)^{k-1}(\tau)}{(1/\chi)^{k}(\tau)}.$$  

(8)
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Inverse iteration with fixed shift performs one step of König’s method. Restarting inverse iteration every \( s \) steps with updated shift given by the current eigenvalue approximation converges with order \( s \) (divided by steps: linearly).
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Symmetric RQI is very pleasant to analyze, likely-wise is two-sided RQI, but unsymmetric RQI (and thus, the QR algorithm) and alternating RQI do not fit into the picture.
The original Rayleigh quotient iteration (Strutt, 1894) with the symmetric Rayleigh quotient and, because of the symmetry, a tridiagonal Hermitean Hessenberg matrix $H_n$, gives the update

$$z_{k+1} = \frac{e_1^T(z_k H_n)^{-1} H_n (z_k H_n)^{-1} e_1}{e_1^T(z_k H_n)^{-1} e_1} = \frac{e_1^T H_n (z_k H_n)^{-2} e_1}{e_1^T(z_k H_n)^{-2} e_1}$$  \hspace{1cm} (9a)

$$= \frac{e_1^T (z_k H_n - z_k H_n)(z_k H_n)^{-2} e_1}{e_1^T(z_k H_n)^{-2} e_1}$$  \hspace{1cm} (9b)

$$= z_k - \frac{e_1^T(z_k H_n)^{-1} e_1}{e_1^T(z_k H_n)^{-2} e_1} = z_k + \frac{[z_k](\chi_{2:n}/\chi)}{[z_k, z_k](\chi_{2:n}/\chi)}$$  \hspace{1cm} (9c)

$$= z_k - \frac{r(z_k)}{r'(z_k)}, \quad r(z) := \frac{\chi(z)}{\chi_{2:n}(z)}.$$  \hspace{1cm} (9d)
**Simplification**

The original Rayleigh quotient iteration (Strutt, 1894) with the symmetric Rayleigh quotient and, because of the symmetry, a tridiagonal Hermitean Hessenberg matrix $H_n$, gives the update

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$$ = \frac{e_1^T(z_k I_n - z_k H_n)(z_k H_n)^{-2} e_1}{e_1^T(z_k H_n)^{-2} e_1} \quad \text{(9b)} $$

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$$ = z_k - \frac{r(z_k)}{r'(z_k)}, \quad r(z) := \frac{\chi(z)}{\chi_{2:n}(z)}. \quad \text{(9d)} $$

This is Newton’s method on the meromorphic function $r$. As the poles of this meromorphic function are the eigenvalues of a submatrix, they interlace by Cauchy’s interlace theorem the roots, which are the eigenvalues.
Symmetric RQI for Hermitean matrices gives the update

\[ z_{k+1} = z_k + \frac{[z_1, z_1, \ldots, z_{k-1}, z_k-1, z_k]}{[z_1, z_1, \ldots, z_{k-1}, z_k-1, z_k]} \left( \chi_{2:n} / \chi \right). \]
Simplification

Symmetric RQI for Hermitean matrices gives the update

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(10)

This update has by a result of Tornheim asymptotically a cubic convergence rate. We have to compute the limit of the real root of the equations

\[ x^k - 2x^{k-1} - 2x^{k-2} - \cdots - 2 = 0, \quad k = 1, \ldots \]
Symmetric RQI for Hermitean matrices gives the update

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This update has by a result of Tornheim asymptotically a cubic convergence rate. We have to compute the limit of the real root of the equations

\[ x^k - 2x^{k-1} - 2x^{k-2} - \cdots - 2 = 0, \quad k = 1, \ldots \]

This is the maximal eigenvalue of a Hessenberg matrix with one in the lower diagonal and two in the last column. The approximate eigenvector of all ones to the approximate eigenvalue 3 gives the backward error \(1/\sqrt{k}\) and the only positive real eigenvalue of the matrix is well separated, the other eigenvalues lie close to a circle of radius one around zero.
The picture changes if we apply the special inverse iteration to a general unreduced Hessenberg matrix, not necessarily Hermitean or symmetric.
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If we take another standard unit vector $e_\ell$ as left vector, we obtain the Opitz-Larkin method applied to the meromorphic function

$$m_\ell(z) = \frac{\chi(z)}{h_{1:\ell-1} \chi_{1:\ell+n}(z)}. \quad (11)$$

The polynomials $\chi_1 + \ell n$ have degree $\deg(\chi_1 + \ell n) = n - \ell$ and leading coefficient one, thus they form a basis of the space of polynomials of degree less $n$. 

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If we take another standard unit vector $e_\ell$ as left vector, we obtain the Opitz-Larkin method applied to the meromorphic function

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If we take an arbitrary left vector $y$, we obtain the Opitz-Larkin method applied to the meromorphic function

$$r(z; y) = \frac{\chi(z)}{\sum_{i=1}^{n} y_i h_{1:i-1}X_{1+i:n}(z)} = \frac{\chi(z)}{p(z; y)}, \quad p(z; y) \in P_{<n}. \hspace{1cm} (12)$$
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The polynomials \( \chi_{1+i:n} \) have degree \( \deg(\chi_{1+i:n}) = n - i \) and leading coefficient one, thus they form a basis of the space of polynomials of degree less \( n \).
The two-sided RQI variant corresponds to a confluent Opitz-Larkin method with double nodes. In this method the left vector determines a polynomial, which is formed as a linear combination of characteristic polynomials of trailing submatrices.
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In single-sided RQI for non-Hermitean matrices, we change the vector \( y \) that determines the denominator polynomial of the rational function

\[
r(z; y) = \frac{\chi(z)}{p(z; y)}
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in every step and apply one step of the Opitz-Larkin method without confluent nodes.
Simplification

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Convergence of $y$ indicates that we might arrive at second order convergence. One multi-shift does not change $y$ compared to several consecutive single shifts. Multiple multi-shifts are locally favourable in the Opitz-Larkin context.
The Jenkins-Traub algorithm is related to a generalized RQI. Thus, it should fit into the Opitz-Larkin framework.
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**First stage:** compute iterates using König’s method at zero,

\[
x_{k+1} = 0 + \frac{[0, \ldots, 0](1/\chi)}{[0, \ldots, 0, 0](1/\chi)} = (k + 1) \frac{(1/\chi)^{(k)}(0)}{(1/\chi)^{(k+1)}(0)}, \quad k = 0, \ldots, p - 1.
\] (13)

**Second stage:** select a fixed shift \(s \in \mathbb{C}\), compute

\[
x_{k+1} = s + \frac{[0, \ldots, 0, s]}{[0, \ldots, 0, 0, s](1/\chi)} = \frac{(1/\chi)^{(k)}(0)}{(1/\chi)^{(k+1)}(0)}, \quad k = p, \ldots, q - 1.
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x_{k+1} = s + \frac{[0, \ldots, 0, s, \ldots, s](1/\chi)}{[0, \ldots, 0, s, \ldots, s, s](1/\chi)}, \quad k = p, \ldots, q - 1. \tag{14}
\]
Third stage: Set the starting value \( z_0 \) to the one obtained by rational interpolation of \( 1/\chi \) at 0 and \( s \), i.e.,

\[
z_0 := x_q = s + \frac{[0, \ldots, 0, s, \ldots, s]}{[0, \ldots, 0, s, \ldots, s, s]}(1/\chi).
\]

(15)

Repeat

\[
z_{k+1} = z_k + \frac{[0, \ldots, 0, s, \ldots, s, z_0, z_1, \ldots, z_{k-1}, z_k]}{[0, \ldots, 0, s, \ldots, s, z_0, z_1, \ldots, z_{k-1}, z_k]}(1/\chi), \quad k = 0, \ldots
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This proves amongst others the well-known fact that stage three of Jenkins-Traub, if it converges, does so with R-order $\phi^2 = \phi + 1 \approx 2.618$. 

... and what about Jenkins-Traub?
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$$z_{k+1} = z_k + \frac{[0, \ldots, 0, s, \ldots, s, z_0, z_1, \ldots, z_{k-1}, z_k](1/\chi)}{[0, \ldots, 0, s, \ldots, s, z_0, z_1, \ldots, z_{k-1}, z_k, z_k](1/\chi)}, \quad k = 0, \ldots$$  \hfill (16)

This proves amongst others the well-known fact that stage three of Jenkins-Traub, if it converges, does so with R-order $\phi^2 = \phi + 1 \approx 2.618$.

Thus, Jenkins-Traub is a special form of Opitz-Larkin with, at first glance, rather strange evaluation scheme. This scheme is natural in view of the companion matrix interpretation given in (Jenkins and Traub, 1970).
We have presented the less well-known Opitz-Larkin method, which is a generalization of König’s method using divided differences.
Conclusions and Outlook

- We have presented the less well-known Opitz-Larkin method, which is a generalization of König’s method using divided differences.
- We have shown that many variants of inverse iteration and Rayleigh quotient iteration correspond to variants of the Opitz-Larkin method on certain rational functions with the characteristic polynomial as the numerator.
- We have indicated why non-symmetric RQI and thus the QR algorithm are not that easily analyzed using this “missing link”.
- We have shown that the well-known Jenkins-Traub method is a special instance of a Opitz-Larkin method.
- Next, we want to take a closer look at the global behaviour of these methods using the Opitz-Larkin framework.
- The local link between one step of Opitz-Larkin and shifts in the QR algorithm should enable a better understanding of multi-shift strategies and the development of new ones.
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