IDR(s)ORes and eigenvalue computations

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joint work with Martin Gutknecht
(work in progress)

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Outline

Krylov subspace methods

- Hessenberg decompositions
- QOR/QMR/Ritz-Galërkin
- OrthoRes-type methods
- LTPM

IDR

- IDR(s)ORes
- Sonneveld pencil and Sonneveld matrix
- Purified pencil
- Deflated pencil and deflated matrix
- BiORes($s, 1$)

Numerical Examples

Hessenberg decompositions

Essential features of Krylov subspace methods can be described by a Hessenberg decomposition

$$\mathbf{A}\mathbf{Q}_n = \mathbf{Q}_{n+1}\underline{\mathbf{H}}_n = \mathbf{Q}_n\mathbf{H}_n + \mathbf{q}_{n+1}h_{n+1,n}\mathbf{e}_n^{\mathsf{T}}. \tag{1}$$

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The matrix $\mathbf{H}_n$ of the perturbed variant will, in general, still be unreduced.

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Generalized Hessenberg decompositions correspond to a skew projection of the pencil $(\mathbf{A}, \mathbf{I})$ to the pencil $(\mathbf{H}_n, \mathbf{U}_n)$ as long as $\mathbf{Q}_{n+1}$ has full rank.

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We show how IDR fits into the LTPM framework.

The prototype IDR(s) (without the recurrences for $\mathbf{x}_n$ and thus already slightly rewritten)

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IDR(s)ORes is based on oblique projections and $s+1$ consecutive multiplications with the same linear factor $\mathbf{I} - \omega_j\mathbf{A}$.

The underlying Hessenberg decomposition

The IDR recurrences of IDR(s)ORes can be summarized by

$$\begin{aligned} \mathbf{v}_{n-1} &:= \mathbf{r}_{n-1} - \nabla \mathbf{R}_{n-s:n-1}\mathbf{c}_n = \mathbf{R}_{n-s-1:n-1}\mathbf{y}_n \\ &= (1-\gamma_s^{(n)})\,\mathbf{r}_{n-1} + \sum_{\ell=1}^{s-1} (\gamma_{s-\ell+1}^{(n)} - \gamma_{s-\ell}^{(n)})\,\mathbf{r}_{n-\ell-1} + \gamma_1^{(n)}\,\mathbf{r}_{n-s-1}\,, \\ \mathbf{r}_n &:= (\mathbf{I} - \omega_j \mathbf{A})\,\mathbf{v}_{n-1}\,. \end{aligned} \tag{7}$$

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compare with the so-called “index functions” (Yeung/Boley, 2005).

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Sonneveld pencil and Sonneveld matrix

The IDR(s)ORes pencil, the so-called Sonneveld pencil $(\mathbf{Y}_n^\circ, \mathbf{Y}_n \mathbf{D}_\omega^{(n)})$, can be depicted by

$$
\begin{pmatrix}
\times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
+ & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & + & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & + & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & + & \times & \times & \times & \times & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & + & \times & \times & \times & \times & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & + & \times & \times & \times & \times & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times & \times & \times & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times
\end{pmatrix},
\begin{pmatrix}
\times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \times & \times & \times & \times & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \times & \times & \times & \times & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \times & \times & \times & \times & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \times & \times & \times & \times & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times & \times & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times & \times \\
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+&\times&\times&\times&\times&\circ&\circ&\circ&\circ&\circ&\circ&\circ\\
\circ&+&\times&\times&\times&\times&\circ&\circ&\circ&\circ&\circ&\circ\\
\circ&\circ&+&\times&\times&\times&\times&\circ&\circ&\circ&\circ&\circ\\
\circ&\circ&\circ&+&\times&\times&\times&\times&\circ&\circ&\circ&\circ\\
\circ&\circ&\circ&\circ&+&\times&\times&\times&\times&\circ&\circ&\circ\\
\circ&\circ&\circ&\circ&\circ&+&\times&\times&\times&\times&\circ&\circ\\
\circ&\circ&\circ&\circ&\circ&\circ&+&\times&\times&\times&\times&\circ\\
\circ&\circ&\circ&\circ&\circ&\circ&\circ&+&\times&\times&\times&\times\\
\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&+&\times&\times&\times\\
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\begin{pmatrix}
\times&\times&\times&\times&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ\\
\circ&\times&\times&\times&\times&\circ&\circ&\circ&\circ&\circ&\circ&\circ\\
\circ&\circ&\times&\times&\times&\times&\circ&\circ&\circ&\circ&\circ&\circ\\
\circ&\circ&\circ&\times&\times&\times&\times&\circ&\circ&\circ&\circ&\circ\\
\circ&\circ&\circ&\circ&\times&\times&\times&\times&\circ&\circ&\circ&\circ\\
\circ&\circ&\circ&\circ&\circ&\times&\times&\times&\times&\circ&\circ&\circ\\
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\circ&\circ&\circ&\circ&\circ&\circ&\circ&\times&\times&\times&\times&\circ\\
\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\times&\times&\times&\times\\
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\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\times&\times\\
\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\times
\end{pmatrix}.$$

The upper triangular matrix $\mathbf{Y}_n\mathbf{D}_\omega^{(n)}$ could be inverted, which results in the Sonneveld matrix, a full unreduced Hessenberg matrix.

Purification

We know the eigenvalues $\approx$ roots of kernel polynomials $1/\omega_j$. We are only interested in the other eigenvalues.

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The purified IDR(s)ORes pencil $(\mathbf{Y}_n^\circ, \mathbf{U}_n\mathbf{D}_\omega^{(n)})$, that has only the remaining eigenvalues and some infinite ones as eigenvalues, can be depicted by

$$
\begin{pmatrix}
\times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
+ & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & + & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & + & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & + & \times & \times & \times & \times & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & + & \times & \times & \times & \times & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & + & \times & \times & \times & \times & \circ & \circ \\
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\circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times
\end{pmatrix},
\begin{pmatrix}
\times & \times & \times & {\color{blue}\circ} & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \times & \times & {\color{blue}\circ} & {\color{blue}\circ} & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \times & {\color{blue}\circ} & {\color{blue}\circ} & {\color{blue}\circ} & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & {\color{blue}\circ} & {\color{blue}\circ} & {\color{blue}\circ} & {\color{blue}\circ} & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \times & \times & \times & {\color{blue}\circ} & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \times & \times & {\color{blue}\circ} & {\color{blue}\circ} & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \times & {\color{blue}\circ} & {\color{blue}\circ} & {\color{blue}\circ} & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & {\color{blue}\circ} & {\color{blue}\circ} & {\color{blue}\circ} & {\color{blue}\circ} & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times & \times & \times & {\color{blue}\circ} \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times & \times & {\color{blue}\circ} \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times & {\color{blue}\circ} \\
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\circ&\circ&+&\times&\times&\times&\times&\circ&\circ&\circ&\circ&\circ\\
\circ&\circ&\circ&+&\times&\times&\times&\times&\circ&\circ&\circ&\circ\\
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\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&+&\times&\times&\times\\
\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&+&\times&\times\\
\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&+&\times
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\begin{pmatrix}
\times&\times&\times&\color{blue}\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ\\
\circ&\times&\times&\color{blue}\circ&\color{blue}\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ\\
\circ&\circ&\times&\color{blue}\circ&\color{blue}\circ&\color{blue}\circ&\circ&\circ&\circ&\circ&\circ&\circ\\
\circ&\circ&\circ&\color{blue}\circ&\color{blue}\circ&\color{blue}\circ&\color{blue}\circ&\circ&\circ&\circ&\circ&\circ\\
\circ&\circ&\circ&\circ&\times&\times&\times&\color{blue}\circ&\circ&\circ&\circ&\circ\\
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\circ&\circ&\circ&\circ&\circ&\circ&\times&\color{blue}\circ&\color{blue}\circ&\color{blue}\circ&\circ&\circ\\
\circ&\circ&\circ&\circ&\circ&\circ&\circ&\color{blue}\circ&\color{blue}\circ&\color{blue}\circ&\color{blue}\circ&\circ\\
\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\times&\times&\times&\color{blue}\circ\\
\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\times&\times&\color{blue}\circ\\
\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\times&\color{blue}\circ\\
\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\circ&\color{blue}\circ
\end{pmatrix}.$$

We get rid of the infinite eigenvalues using a change of basis (Gauß/Schur).

Gaussian elimination

The deflated purified IDR(s)ORes pencil, after the elimination step $(\mathbf{Y}_n^\circ \mathbf{G}_n, \mathbf{U}_n \mathbf{D}_\omega^{(n)})$, can be depicted by

$$
\begin{pmatrix}
\times & \times & \times & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ \\
+ & \times & \times & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ \\
\circ & + & \times & \times & \times & \times & \times & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & + & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & + & + & \times & \times & \times & \times & \times & \times & \times & \circ \\
\circ & \circ & \circ & \circ & + & \times & \times & \times & \times & \times & \times & \circ \\
\circ & \circ & \circ & \circ & \circ & + & \times & \times & \times & \times & \times & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & + & + & \times & \times & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & +
\end{pmatrix},
\begin{pmatrix}
\times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \times & \times & \times & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \times & \times & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \times & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times & \times & \times & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times & \times & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ
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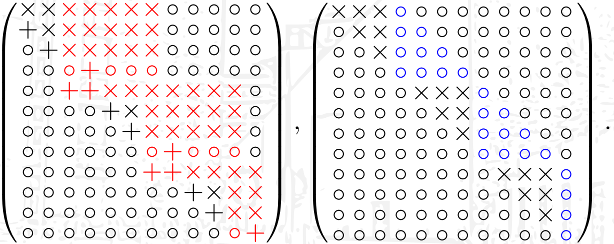

Using Laplace expansion of the determinant of $z\mathbf{U}_n\mathbf{D}_\omega^{(n)} - \mathbf{Y}_n^\circ \mathbf{G}_n$ we can get rid of the trivial constant factors corresponding to infinite eigenvalues. This amounts to a deflation.

Deflation

The deflated purified IDR(s)ORes pencil, after the deflation step $(D(\mathbf{Y}_n^\circ \mathbf{G}_n), D(\mathbf{U}_n \mathbf{D}_\omega^{(n)}))$, can be depicted by

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times & \circ & \circ & \circ \\ + & \times & \times & \times & \times & \times & \circ & \circ & \circ \\ \circ & + & \times & \times & \times & \times & \circ & \circ & \circ \\ \circ & \circ & + & \times & \times & \times & \times & \times & \times \\ \circ & \circ & \circ & + & \times & \times & \times & \times & \times \\ \circ & \circ & \circ & \circ & + & \times & \times & \times & \times \\ \circ & \circ & \circ & \circ & \circ & + & \times & \times & \times \\ \circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times \end{pmatrix}, \begin{pmatrix} \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \times & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \times & \times & \times & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \times & \times & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \times & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \times & \times & \times \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times & \times \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times \end{pmatrix}.$$

Deflation

The deflated purified IDR(s)ORes pencil, after the deflation step $(D(\mathbf{Y}_n^\circ \mathbf{G}_n), D(\mathbf{U}_n \mathbf{D}_\omega^{(n)}))$, can be depicted by

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times & \circ & \circ & \circ \\ + & \times & \times & \times & \times & \times & \circ & \circ & \circ \\ \circ & + & \times & \times & \times & \times & \circ & \circ & \circ \\ \circ & \circ & + & \times & \times & \times & \times & \times & \times \\ \circ & \circ & \circ & + & \times & \times & \times & \times & \times \\ \circ & \circ & \circ & \circ & + & \times & \times & \times & \times \\ \circ & \circ & \circ & \circ & \circ & + & \times & \times & \times \\ \circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times \end{pmatrix}, \begin{pmatrix} \times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \times & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \times & \times & \times & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \times & \times & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \times & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \times & \times & \times \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times & \times \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times \end{pmatrix}.$$

Here, D is an deflation operator that removes every $s+1$th column and row from the matrix the operator is applied to.

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$$\begin{pmatrix}
\times & \times & \times & \times & \times & \times & \circ & \circ & \circ \\
+ & \times & \times & \times & \times & \times & \circ & \circ & \circ \\
\circ & + & \times & \times & \times & \times & \circ & \circ & \circ \\
\circ & \circ & + & \times & \times & \times & \times & \times & \times \\
\circ & \circ & \circ & + & \times & \times & \times & \times & \times \\
\circ & \circ & \circ & \circ & + & \times & \times & \times & \times \\
\circ & \circ & \circ & \circ & \circ & + & \times & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times
\end{pmatrix}, \quad
\begin{pmatrix}
\times & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \times & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \times & \times & \times & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \times & \times & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \times & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \times & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \times & \times \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \times
\end{pmatrix}.$$

Here, D is an deflation operator that removes every $s+1$th column and row from the matrix the operator is applied to.

The block-diagonal matrix $D(\mathbf{U}_n \mathbf{D}_\omega^{(n)})$ has invertible upper triangular blocks and can be inverted to expose the underlying Lanczos process.

A Lanczos process with multiple left-hand sides

Inverting the block-diagonal matrix $D(\mathbf{U}_n\mathbf{D}_\omega^{(n)})$ gives an algebraic eigenvalue problem with a block-tridiagonal unreduced upper Hessenberg matrix

$$\mathbf{L}_n := D(\mathbf{Y}_n^\circ \mathbf{G}_n) \cdot D(\mathbf{U}_n\mathbf{D}_\omega^{(n)}))^{-1} = \begin{pmatrix} \times & \times & \times & \times & \times & \times & \circ & \circ & \circ \\ + & \times & \times & \times & \times & \times & \circ & \circ & \circ \\ \circ & + & \times & \times & \times & \times & \circ & \circ & \circ \\ \circ & \circ & + & \times & \times & \times & \times & \times & \times \\ \circ & \circ & \circ & + & \times & \times & \times & \times & \times \\ \circ & \circ & \circ & \circ & + & \times & \times & \times & \times \\ \circ & \circ & \circ & \circ & \circ & + & \times & \times & \times \\ \circ & \circ & \circ & \circ & \circ & \circ & + & \times & \times \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & + & \times \end{pmatrix}.$$

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This is the matrix of the underlying BiORes(s, 1) process.

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This is the matrix of the underlying BiORes($s, 1$) process.

This matrix (in the extended version) satisfies

$$\mathbf{A}\mathbf{Q}_n = \mathbf{Q}_{n+1}\underline{\mathbf{L}}_n,$$

where the reduced residuals $\mathbf{q}_{js+k}$, $k = 0, \ldots, s-1$, $j = 0, 1, \ldots$, with $\Omega_0(z) \equiv 1$ and $\Omega_j(z) = \prod_{k=1}^{j}(1 - \omega_k z)$ are given by

$$\Omega_j(\mathbf{A})\mathbf{q}_{js+k} = \mathbf{r}_{j(s+1)+k}.$$

A Lanczos process with multiple left-hand sides

The reduced residuals are defined by

$$\Omega_j(\mathbf{A})\mathbf{q}_{js+k} = \mathbf{r}_{j(s+1)+k} = (\mathbf{I} - \omega_j\mathbf{A})\mathbf{v}_{j(s+1)+k-1}$$

and every $\mathbf{v}_{j(s+1)+k-1}$ is orthogonal to $\mathbf{P}$.

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Thus, $\mathbf{q}_{js+k} \perp \Omega_{j-1}(\mathbf{A}^{\mathsf{H}})\mathbf{P}$.

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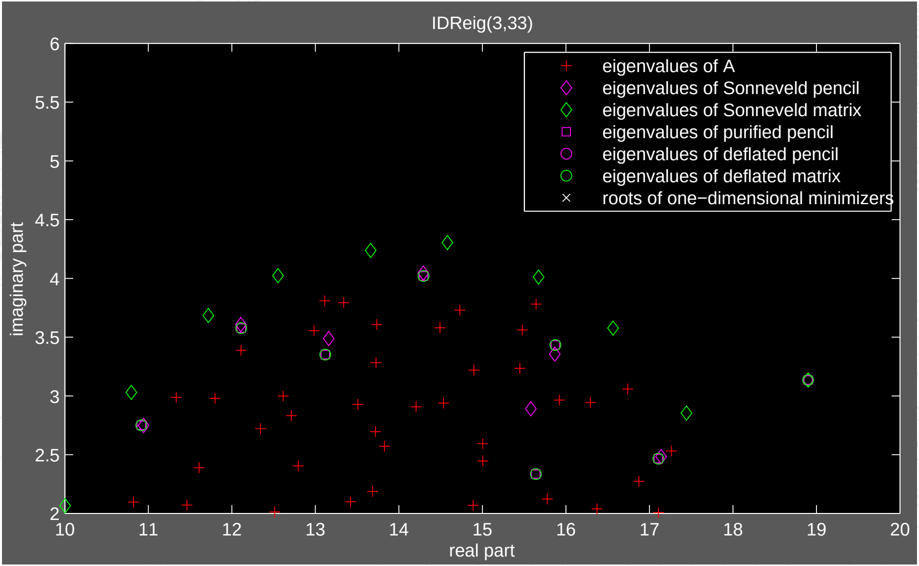
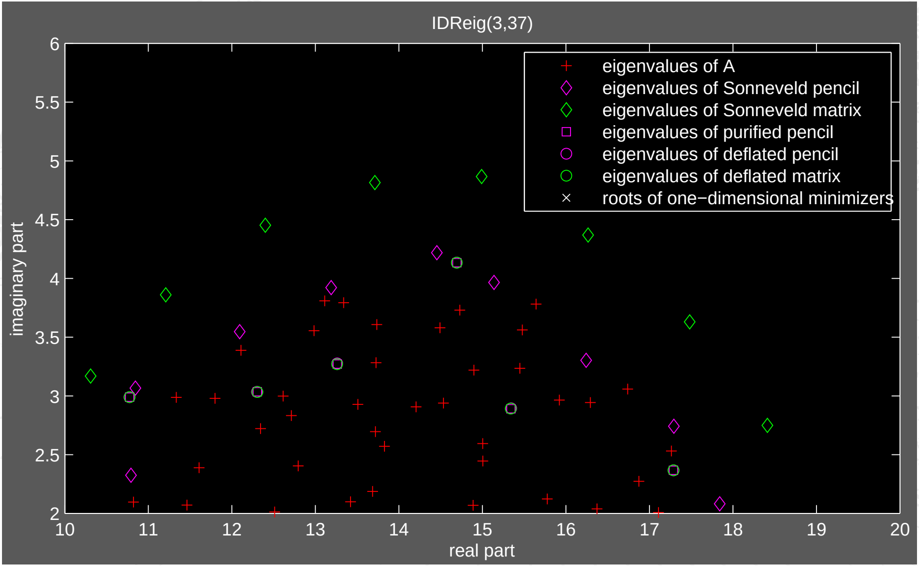
Thus, $\mathbf{q}_{js+k} \perp \Omega_{j-1}(\mathbf{A}^\mathsf{H})\mathbf{P}$.

Using induction one can prove that $\mathbf{q}_{js+k} \perp \mathcal{K}_j(\mathbf{A}^\mathsf{H}, \mathbf{P})$; thus, this is a two-sided Lanczos process with s left and one right starting vectors.

Selected examples for $s = 3$



Selected examples for $s = 3$



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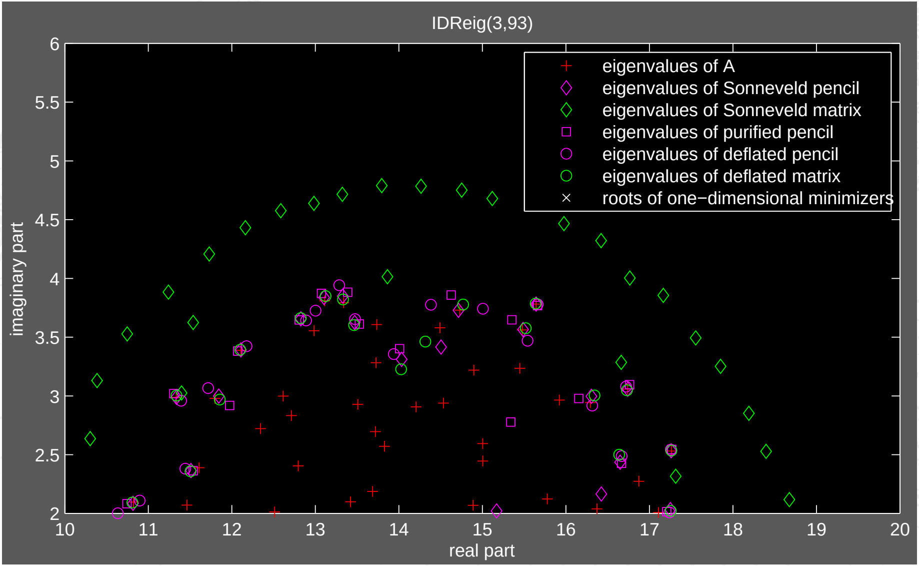

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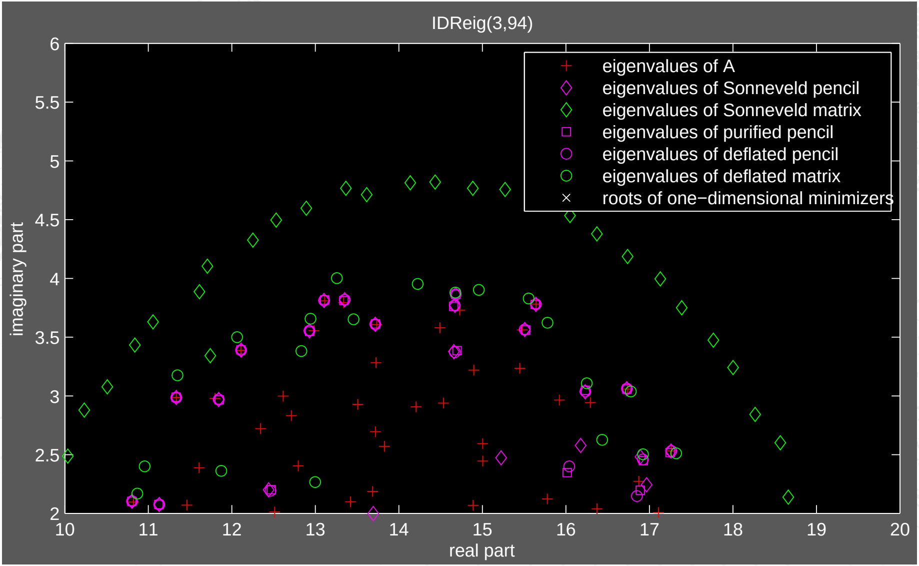

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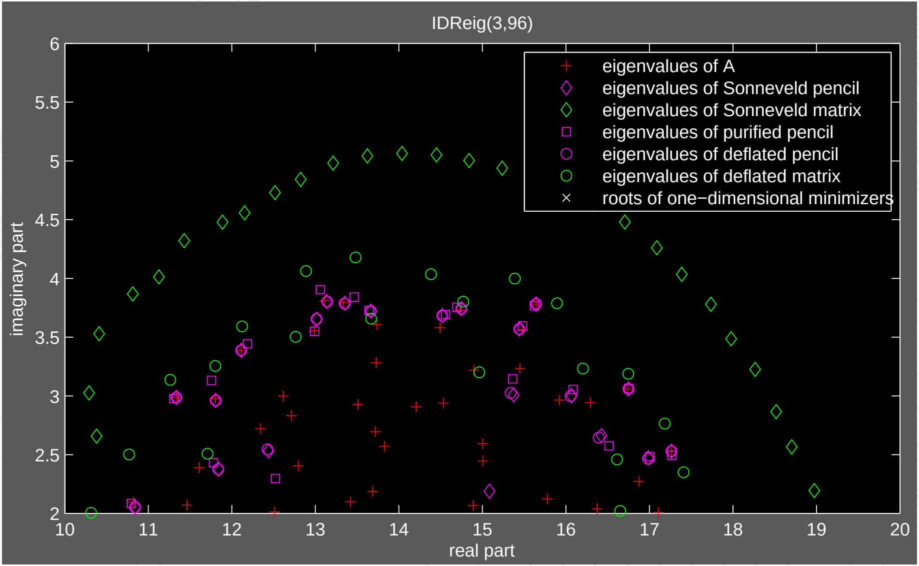

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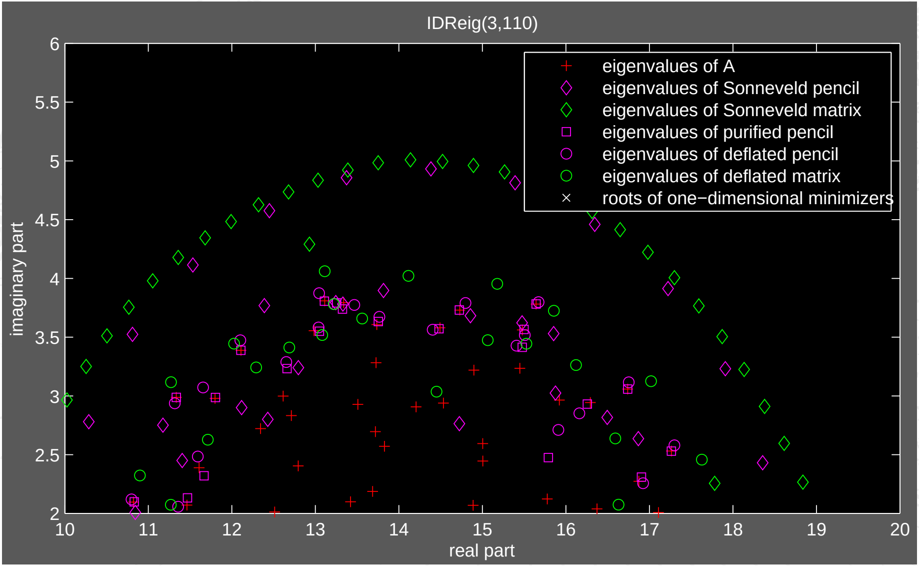

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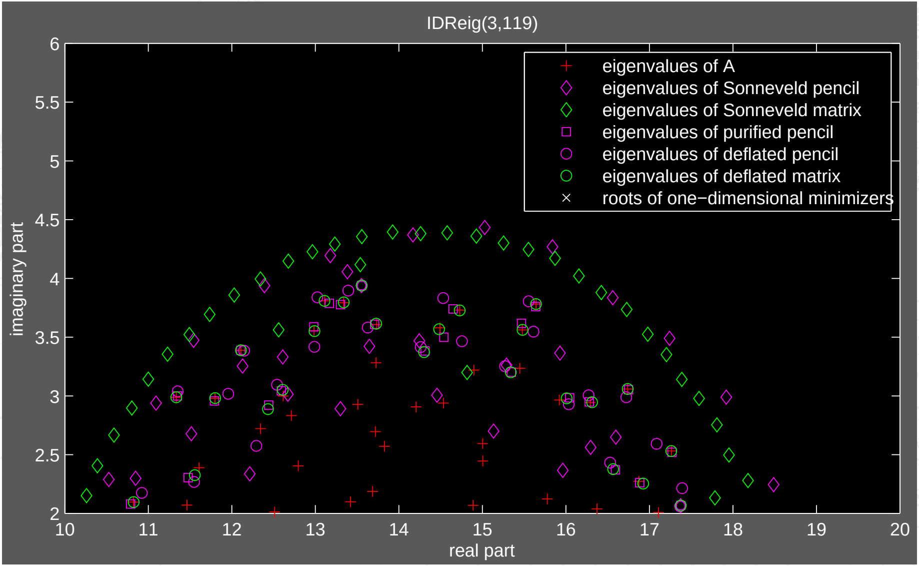

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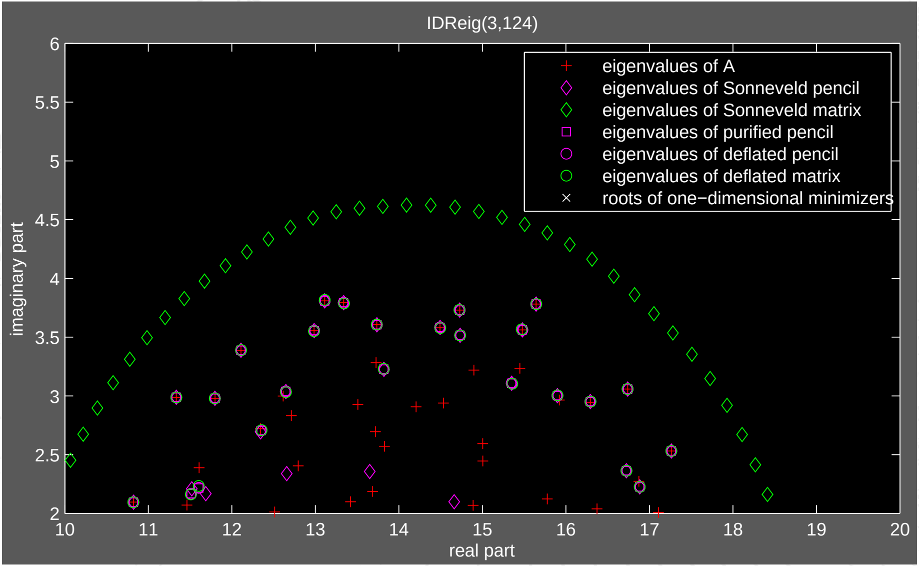

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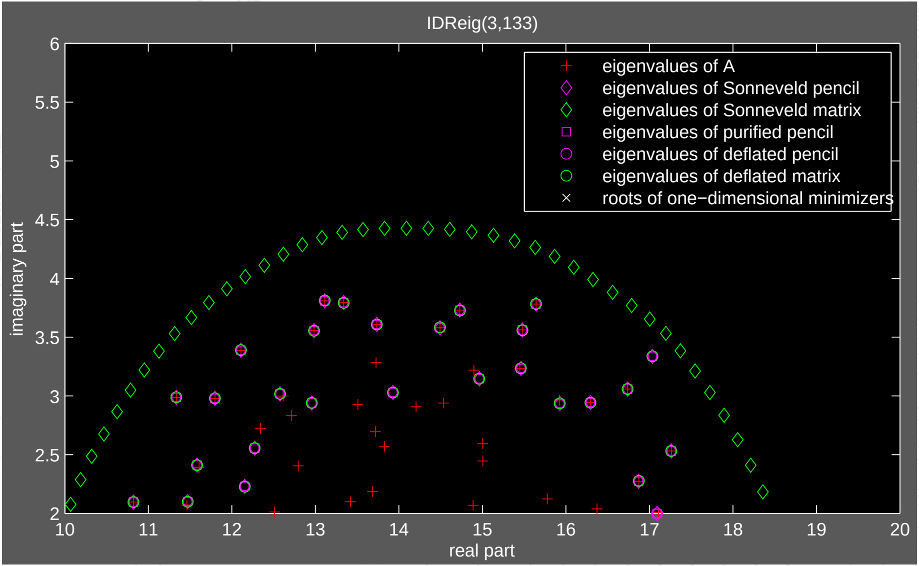

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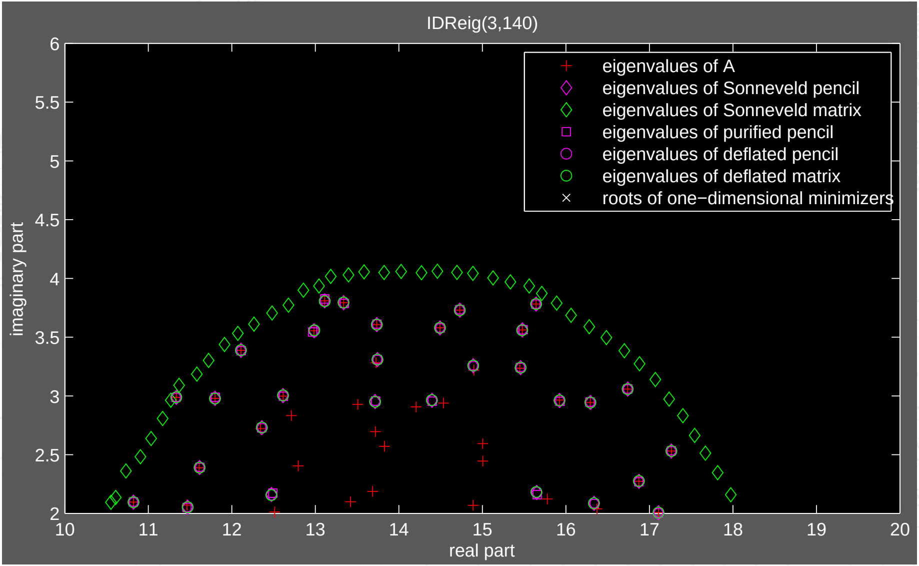

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