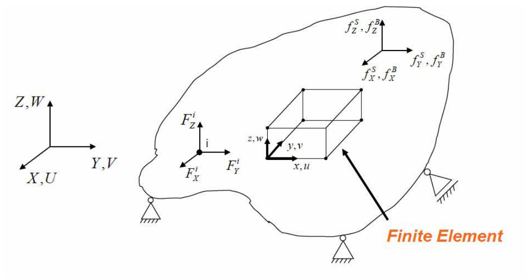
Finite Elements Methods

Formulary for Prof. Estorff's exam

Finite Element Method in General



One wants to obtain the equilibrium eqautions for the body, discretized by finite elements in the form

$M \cdot \ddot{U} + C \cdot \dot{U} + K \cdot U = R$

Displacement of the nodes:

$U = \left[\begin{array}{cccc} U_1 & U_2 & \ldots & U_n \end{array} \right]^T$ $\quad$ n: degrees of freedom

Displacement within the element m:

$u^{(m)}(x,y,z) = H^{(m)}(x,y,z)\hat{U}$

$H^{(m)}$: Displacement interpolation matrix

Strain inside the element m:

$\epsilon^{(m)}(x,y,z) = B^{(m)}(x,y,z)\hat{U}$

$B^{(m)}$: Verzerrungs-Verschiebungs-Matrix

Play around with those matrices and from the priciple of virtual displacement $\int\limits_V \bar{\epsilon}^T \tau \, dV = \int\limits_V \bar{U}^T f^B dV + \int\limits_S \bar{U}^{ST} f^S \, dS + \sum\limits_i \bar{U}^{iT} F^i$ it follows:

System of equations (stationary case $KU = R$)

$$\left[\sum_m \int_{V^{(m)}} B^{(m)T} C^{(m)} B^{(m)} \, dV^{(m)} \right] \hat{U} = \sum_m \int_{V^{(m)}} H^{(m)T} f^{B(m)} \, dV^{(m)}$$
$$+ \sum_m \int_{S^{(m)}} H^{S(m)T} f^{S(m)} \, dS^{(m)} + \sum_m \int_{V^{(m)}} B^{(m)T} \tau^{I(m)} \, dV^{(m)} + F$$

From that one can obtain a formular to calculate each of the matrices in the equlibrium equasions:

Stiffness matrix: $K = \sum_m \int_{V^{(m)}} B^{(m)T} C^{(m)} B^{(m)} dV^{(m)} = \sum K^{(m)}$

Volume forces: $R_B = \sum_m \int_{V^{(m)}} H^{(m)T} f^{B(m)} \, dV^{(m)} = \sum_m R_B^{(m)}$

Surface forces: $R_S = \sum_m \int_{S^{(m)}} H^{S(m)T} f^{S(m)} \, dS^{(m)} = \sum_m R_S^{(m)}$

Initial stresses: $R_I = \sum_m \int_{v^{(m)}} B^{(m)T} \tau^{I(m)} \, dV^{(m)} = \sum_m R_I^{(m)}$

Single forces: $R_C = F$

With d'Alembert Principle the matrices for the time dependent case follow:

Mass matrix: $M = \sum_m \int_{V^{(m)}} \rho^{(m)} H^{(m)T} H^{(m)} \, dV^{(m)} = \sum_m M^{(m)}$

Damping matrix: $C = \sum_m \int_{V^{(m)}} \kappa^{(m)} H^{(m)T} H^{(m)} \, dV^{(m)} = \sum_m C^{(m)}$

According to the choosen elements one has to concidere only certain components of stress, strain and displacement:

TABLE 4.2 *Corresponding kinematic and static variables in various problems*

Problem	Displacement components	Strain vector ϵ^T	Stress vector τ^T
Bar	u	$[\epsilon_{xx}]$	$[\tau_{xx}]$
Beam	w	$[\kappa_{xx}]$	$[M_{xx}]$
Plane stress	u, v	$[\epsilon_{xx}\ \epsilon_{yy}\ \gamma_{xy}]$	$[\tau_{xx}\ \tau_{yy}\ \tau_{xy}]$
Plane strain	u, v	$[\epsilon_{xx}\ \epsilon_{yy}\ \gamma_{xy}]$	$[\tau_{xx}\ \tau_{yy}\ \tau_{xy}]$
Axisymmetric	u, v	$[\epsilon_{xx}\ \epsilon_{yy}\ \gamma_{xy}\ \epsilon_{zz}]$	$[\tau_{xx}\ \tau_{yy}\ \tau_{xy}\ \tau_{zz}]$
Three-dimensional	u, v, w	$[\epsilon_{xx}\ \epsilon_{yy}\ \epsilon_{zz}\ \gamma_{xy}\ \gamma_{yz}\ \gamma_{zx}]$	$[\tau_{xx}\ \tau_{yy}\ \tau_{zz}\ \tau_{xy}\ \tau_{yz}\ \tau_{zx}]$
Plate bending	w	$[\kappa_{xx}\ \kappa_{yy}\ \kappa_{xy}]$	$[M_{xx} M_{yy} M_{xy}]$

Notation: $\epsilon_{xx} = \frac{\partial u}{\partial x}$, $\epsilon_{yy} = \frac{\partial v}{\partial y}$, $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, $\ldots$, $\kappa_{xx} = \frac{\partial^2 w}{\partial x^2}$, $\kappa_{yy} = \frac{\partial^2 w}{\partial y^2}$, $\kappa_{xy} = 2\frac{\partial^2 w}{\partial x \, \partial y}$.

(Isoparametric) Truss Elements

Displacement of the nodes:

Global: $\hat{U} = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & \dots & v_n \end{bmatrix}^T$

Local: $\tilde{U} = \begin{bmatrix} \tilde{u}_1 & \tilde{v}_1 & \tilde{u}_2 & \tilde{v}_2 & \dots & \tilde{v}_n \end{bmatrix}^T$ $\quad n$: number of nodes in element

$\hat{U}$ can be linked with the overall node displacement vector of the system (has the same orientation).

Interpolation:

Coordinates: $\tilde{x}(r) = \sum h_i \tilde{x}_i$ Displacements: $\tilde{u}(r) = \sum h_i \tilde{u}_i$

In the following, an element with 2 nodes is considered.

Displacement interpolation matrix:

$$\tilde{H} = \begin{bmatrix} h_1 & 0 & h_2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1-r) & 0 & \frac{1}{2}(1+r) & 0 \end{bmatrix} \text{ with } \begin{array}{c} x(r) = \tilde{H} \cdot \tilde{X} \\ u(r) = \tilde{H} \cdot \tilde{U} \end{array}$$

Setting up the strain displacement matrix:

$$\begin{array}{l} \epsilon_x = \frac{du}{dx} \\ \\ \epsilon = \tilde{B} \cdot \tilde{U} \end{array} \Longrightarrow \tilde{B} = \begin{bmatrix} h_{1x} & 0 & h_{2x} & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} & 0 & \frac{1}{L} & 0 \end{bmatrix}$$

because $h_{ix} = \frac{\partial h_i}{\partial x} = \frac{\partial h_i}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial h_i}{\partial r}\left(\frac{\partial x}{\partial r}\right)^{-1} = \frac{\partial h_i}{\partial r}\left(\frac{L}{2}\right)^{-1}$

Setting up the stiffness matrix:

$$\tilde{K} = \int_V \tilde{B}^T C \tilde{B}\, dV = AE \int_0^L \tilde{B}^T \tilde{B}\, dx$$ with A: Cross-sectional area of truss

It follows: $$\tilde{K} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Transformed to global directions:

$$K = T^T \tilde{K} T = \frac{EA}{L} \begin{bmatrix} cos^2\alpha & cos\alpha\, sin\alpha & -cos\alpha\, cos\beta & -cos\alpha\, sin\beta \\ cos\alpha\, sin\alpha & sin^2\alpha & -sin\alpha\, cos\beta & -sin\alpha\, sin\beta \\ -cos\alpha\, cos\beta & -sin\alpha\, cos\beta & cos^2\beta & sin\beta\, cos\beta \\ -cos\alpha\, sin\beta & -sin\alpha\, sin\beta & sin\beta\, cos\beta & sin^2\beta \end{bmatrix}$$

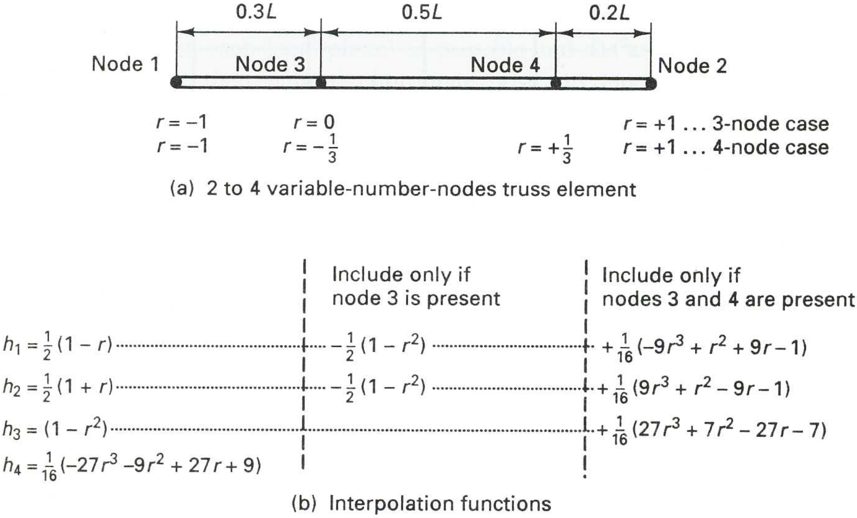
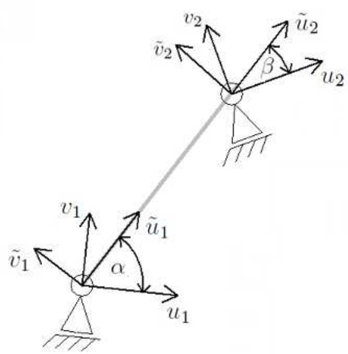


Figure 5.3 Interpolation functions of two to four variable-number-nodes one-dimensional element

Transformation matrix

$\tilde{u} = Tu$

$$T = \begin{bmatrix} cos\alpha & sin\alpha & 0 & 0 \\ -sin\alpha & cos\alpha & 0 & 0 \\ 0 & 0 & cos\beta & sin\beta \\ 0 & 0 & -sin\beta & cos\beta \end{bmatrix}$$



Related transformations:

$H = \tilde{H}T$

$M = T^T \tilde{M} T$

$R_B = T^T \tilde{R}_B$

$R_S = T^T \tilde{R}_S$

$R_I = T^T \tilde{R}_I$

Isoparametric Plate Elements

Displacement of the nodes:

$\hat{U} = \begin{bmatrix} u_1 & \dots & u_n & v_1 & \dots & v_n \end{bmatrix}^T$ n: number of nodes in element

$\hat{U}$ can be linked with the overall node displacement vector of the system.

Interpolation:

Coordinates: $x(r,s) = \sum h_i x_i$; $y(r,s) = \sum h_i y_i$

Displacements: $u(r,s) = \sum h_i u_i$; $v(r,s) = \sum h_i v_i$

In the following formulars, an element with 4 nodes is concidered.

Displacement interpolation matrix:

$$H = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 \end{bmatrix} \text{ with } \begin{array}{c} X(r,s) = H \cdot \hat{X} \\ U(r,s) = H \cdot \hat{U} \end{array}$$

Match H with the vector of displacements of all nodes U to obtain the global stiffness matric $H^{(m)}$ for the element.

Setting up the strain displacement matrix:

$$\begin{array}{c} \epsilon_x = \frac{du}{dx} \quad \epsilon_y = \frac{dv}{dy} \\ \epsilon_{xy} = \frac{du}{dy} + \frac{dv}{dx} \\ \epsilon = B \cdot \hat{U} \end{array} \quad \Longrightarrow B = \begin{bmatrix} h_{1x} & h_{2x} & h_{3x} & h_{4x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_{1y} & h_{2y} & h_{3y} & h_{4y} \\ h_{1y} & h_{2y} & h_{3y} & h_{4y} & h_{1x} & h_{2x} & h_{3x} & h_{4x} \end{bmatrix}$$

Match B with the vector of displacements of all nodes U to obtain the global stiffness matric $B^{(m)}$ for the element.

Calculating derivatives of h_i:

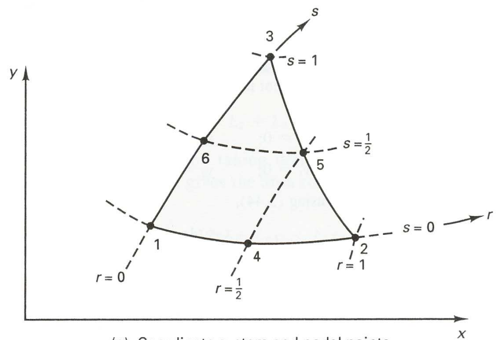
$$\begin{bmatrix} h_{ix} \\ h_{iy} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_i}{\partial x} \\ \frac{\partial h_i}{\partial y} \end{bmatrix} = J^{-1} \cdot \begin{bmatrix} \frac{\partial h_i}{\partial r} \\ \frac{\partial h_i}{\partial s} \end{bmatrix} \text{ with } J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}$$

Inverse 4x4 matrix: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Setting up the stiffness matrix:

$K = \int_V B^T C B \, dV = t \cdot \int_{-1}^{1} \int_{-1}^{1} B^T C B \, dr \, ds$ t: thickness of the element

Match K with the vector of displacements of all nodes U to obtain the global stiffness matric $K^{(m)}$ for the element.



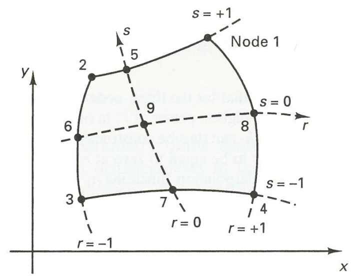
(a) Coordinate system and nodal points

Include only if node i is defined

		$i = 4$	$i = 5$	$i = 6$
$h_1 =$	$1 - r - s$	$-\frac{1}{2}h_4$		$-\frac{1}{2}h_6$
$h_2 =$	r	$-\frac{1}{2}h_4$	$-\frac{1}{2}h_5$	
$h_3 =$	s		$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$
$h_4 =$	$4r(1 - r - s)$			
$h_5 =$	$4rs$			
$h_6 =$	$4s(1 - r - s)$			

(b) Interpolation functions

Figure 5.11 Interpolation functions of three to six variable-number-nodes two-dimensional triangle



(a) 4 to 9 variable-number-nodes two-dimensional element

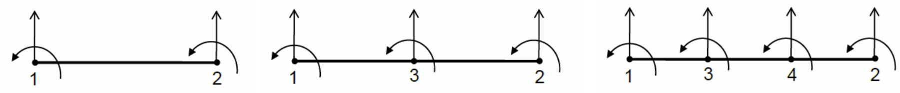
Include only if node i is defined

		$i = 5$	$i = 6$	$i = 7$	$i = 8$	$i = 9$
$h_1 =$	$\frac{1}{4}(1 + r)(1 + s)$	$-\frac{1}{2}h_5$			$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_2 =$	$\frac{1}{4}(1 - r)(1 + s)$	$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$			$-\frac{1}{4}h_9$
$h_3 =$	$\frac{1}{4}(1 - r)(1 - s)$		$-\frac{1}{2}h_6$	$-\frac{1}{2}h_7$		$-\frac{1}{4}h_9$
$h_4 =$	$\frac{1}{4}(1 + r)(1 - s)$			$-\frac{1}{2}h_7$	$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_5 =$	$\frac{1}{2}(1 - r^2)(1 + s)$					$-\frac{1}{2}h_9$
$h_6 =$	$\frac{1}{2}(1 - s^2)(1 - r)$					$-\frac{1}{2}h_9$
$h_7 =$	$\frac{1}{2}(1 - r^2)(1 - s)$					$-\frac{1}{2}h_9$
$h_8 =$	$\frac{1}{2}(1 - s^2)(1 + r)$					$-\frac{1}{2}h_9$
$h_9 =$	$(1 - r^2)(1 - s^2)$					

(b) Interpolation functions

Figure 5.4 Interpolation functions of four to nine variable-number-nodes two-dimensional element

Beam Elements



Displacement of the nodes:

$\hat{U} = \left[\begin{array}{cccc} w_1 & \varphi_1 & w_2 & \varphi_2 \end{array}\right]^T$ for an element with 2 nodes

Displacement interpolation matrix (Hermite Beam):

$w(x) = H\hat{U}$

$$H = \left[\begin{array}{cccc} 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3} & x - 2\frac{x^2}{L} + \frac{x^3}{L^2} & 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3} & -\frac{x^2}{L} + \frac{x^3}{L^2} \end{array}\right]$$

$$\Longrightarrow K = EI \left[\begin{array}{cccc} \frac{12}{L^3} & & & \\ \frac{6}{L^2} & \frac{4}{L} & & \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{array}\right]$$

For different H the strain displacement matrix B has to be determined first to calculate $K = \int_V B^T C B \, dV$.

Isoparametric:

Displacement of the nodes:

$\hat{U} = \left[\begin{array}{cccccc} w_1 & \dots & w_q & \varphi_1 & \dots & \varphi_q \end{array}\right]^T$ q: number of nodes in element

Interpolation:

$w(r) = \sum h_i w_i \qquad \varphi(r) = \sum h_i \varphi_i$

Use the interpolation functions for a 1D element.

For displacement interpolation matrix H and strain interpolation matrix B concider displacement and rotation seperatly:

$H_w = \left[\begin{array}{cccccc} h_1 & \dots & h_q & 0 & \dots & 0 \end{array}\right] \qquad H_\varphi = \left[\begin{array}{cccccc} 0 & \dots & 0 & h_1 & \dots & h_q \end{array}\right]$

$B_w = J^{-1} \left[\begin{array}{cccccc} \frac{\partial h_1}{\partial r} & \dots & \frac{\partial h_q}{\partial r} & 0 & \dots & 0 \end{array}\right] \quad B_\varphi = J^{-1} \left[\begin{array}{cccccc} 0 & \dots & 0 & \frac{\partial h_1}{\partial r} & \dots & \frac{h_q}{\partial r} \end{array}\right]$

with the Jacobian $J = \frac{\partial x}{\partial r}$

Material Matrices

TABLE 4.3 *Generalized stress-strain matrices for isotropic materials and the problems in Table 4.2*

Problem	Material matrix **C**
Bar	E
Beam	EI
Plane stress	$\dfrac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$
Plane strain	$\dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 \\ \dfrac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} \end{bmatrix}$
Axisymmetric	$\dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 & \dfrac{\nu}{1-\nu} \\ \dfrac{\nu}{1-\nu} & 1 & 0 & \dfrac{\nu}{1-\nu} \\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} & 0 \\ \dfrac{\nu}{1-\nu} & \dfrac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}$
Three-dimensional	$\dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & \dfrac{\nu}{1-\nu} & & & \\ \dfrac{\nu}{1-\nu} & 1 & \dfrac{\nu}{1-\nu} & & & \\ \dfrac{\nu}{1-\nu} & \dfrac{\nu}{1-\nu} & 1 & & & \\ & & & \dfrac{1-2\nu}{2(1-\nu)} & & \\ & & & & \dfrac{1-2\nu}{2(1-\nu)} & \\ & & & & & \dfrac{1-2\nu}{2(1-\nu)} \end{bmatrix}$ Elements not shown are zeros
Plate bending	$\dfrac{Eh^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$

Notation: E = Young's modulus, ν = Poisson's ratio, h = thickness of plate, I = moment of inertia

Calculating Loads

In general $R_S = \int\limits_S H^{ST} q(x)\, dS$ for distributed loads $q(x)$.

In the following a load acting between note 1 and 2 of a 4-node-isoparametric element where $s = const = 1$ is considered.

Set up H^S matrix

Consider only the displacements of the two nodes contained in the surface the load is working on:

$$\begin{bmatrix} u_S \\ v_S \end{bmatrix} = H^S \hat{U}^S \quad \text{mit } \hat{U}^S = \begin{bmatrix} u_1 & u_2 & v_1 & v_2 \end{bmatrix}$$

$$H^S = \begin{bmatrix} \frac{1}{2}(1+r) & \frac{1}{2}(1-r) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(1+r) & \frac{1}{2}(1-r) \end{bmatrix}$$

Integrate

Rewrite $q(x)$ in local coordinates to obtain $q(r)$ and integrate over the surface:

$$R = \begin{bmatrix} R_{x1} & R_{x2} & R_{y1} & R_{y2} \end{bmatrix}^T = t \cdot \int\limits_{-1}^{1} H^{ST} q(r)\, det(J^S)\, dr$$

with $q(r) = \begin{bmatrix} q_x(r) \\ q_y(r) \end{bmatrix} \hat{=} f^S$ and t: thickness of the plate

Jabobian matrix

Since $s = const$ here, the Jacobian degenerates to $J^S = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \end{bmatrix}$.

Because J^S is not quadratic anymore, one has to use the Gramian determinant:

$$det(J^S) = \sqrt{J^S J^{ST}} = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2}$$

Load in one direction (after the other)

For a vertical load with $q_x(x) = 0$ only the vertical displacement and forces have to be concidered:

$$v_S = H^S \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \implies H^S = \begin{bmatrix} h_1 & h_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1+r) & \frac{1}{2}(1-r) \end{bmatrix}$$

$$R = \begin{bmatrix} R_{y1} \\ R_{y2} \end{bmatrix} = t \cdot \int\limits_{-1}^{1} H^{ST} q_y(r)\, det(J)\, dr$$

Always try set up single forces that representate the distributed load first!

Setting up Mass Matrices

$M^{(m)} = \int\limits_{V^{(m)}} \rho^{(m)} H^{(m)T} H^{(m)} \, dV^{(m)}$

Consistent mass matrix:

1D: $M^{(m)} = \int\limits_{0}^{L} A(x) \rho^{(m)} H^{(m)T} H^{(m)} \, dx = \int\limits_{-1}^{1} A(r) \rho^{(m)} H^{(m)T} H^{(m)} det(J) \, dr$

2D: $M^{(m)} = \int\limits_{-1}^{1} \int\limits_{-1}^{1} t(x) \rho^{(m)} H^{(m)T} H^{(m)} det(J) \, dr \, ds$

Lumped mass matrix:

$M^{(m)} = \begin{bmatrix} \frac{m}{n} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{m}{n} \end{bmatrix}$ with m: Mass of the element; n: Number of nodes in element

Calculating Stresses

In general: $\sigma = C \cdot BU$

Truss: $\sigma = C\epsilon = E\frac{\Delta L}{L}$ with $\Delta L = \tilde{u}_2 - \tilde{u}_1$

Plate (at $s = s^*$, $r = r^*$): $\sigma = C \cdot B \mid_{r^*, s^*} U$

Jacobian Matrix

Transformation of coordinates / chain rule /...:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{bmatrix} \text{ so that } \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{bmatrix} = J \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Used to calculate:

$$\begin{bmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial h}{\partial r} \\ \frac{\partial h}{\partial s} \\ \frac{\partial h}{\partial t} \end{bmatrix} \text{ and } \int_V \ldots \, dV = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \ldots \, det(J)\, dr\, ds\, dt$$

Rules for Variational Operator

$$\delta \frac{du}{dx} = \frac{d}{dx}\delta u \qquad \delta \int_a^b f(x)\, dx = \int_a^b \delta f(s)\, dx$$

Integration by Parts

$$\int_a^b u'(x)v(x)\, dx = [u(x)v(x)]_a^b - \int_a^b u(x)v'(x)\, dx$$

Variation of the Total Potential

Total potential: $\Pi = U - W$

U : Strain energy $\quad W$: Potential of external loads

Obtain natural boundary conditions and partial differential equation by solving $\delta\Pi = 0$, where Π is the (hopefully given) potential of the system.

Ritz Method

Insert an approximation function for displacement with coefficients a_1, a_2,a_3, ... and solve $\frac{\partial \Pi}{\partial a_1}$, $\frac{\partial \Pi}{\partial a_2}$, ...

Note that $\frac{\partial}{\partial a_1}\int_a^b f(a_i)\, dx = \int_a^b \frac{\partial}{\partial a_1} f(a_i)\, dx$.

More stuff...

Gauss Integration

$K = \int_V F(r,s,t)\, dr\, ds\, dt \quad \text{mit} \quad F(r,s,t) = B^T C B\, det(J)$

Gauss:

$\int_a^b F(r)\, dr = \alpha_1 F(r_1) + \alpha_2 F(r_2) + ... + \alpha_n F(r_n)$

Polynomials up to the order $(2n-1)$ can be integrated axactly.

Example

The integral $\int_{-1}^{1} F(r)\, dr$ shall be approximated using

just one approximation point:

$\int_{-1}^{1} F(r)\, dr = 2 \cdot F(0)$

two approximation points:

$\int_{-1}^{1} F(r)\, dr = 1 \cdot F(-0.57735) + 1 \cdot F(0.57735)$

TABLE 5.6 *Sampling points and weights in Gauss-Legendre numerical integration (interval −1 to +1)*

n	r_i			α_i		
1	0.	(15 zeros)		2.	(15 zeros)	
2	±0.57735	02691	89626	1.00000	00000	00000
3	±0.77459	66692	41483	0.55555	55555	55556
	0.00000	00000	00000	0.88888	88888	88889
4	±0.86113	63115	94053	0.34785	48451	37454
	±0.33998	10435	84856	0.65214	51548	62546
5	±0.90617	98459	38664	0.23692	68850	56189
	±0.53846	93101	05683	0.47862	86704	99366
	0.00000	00000	00000	0.56888	88888	88889
6	±0.93246	95142	03152	0.17132	44923	79170
	±0.66120	93864	66265	0.36076	15730	48139
	±0.23861	91860	83197	0.46791	39345	72691

Newton-Cotes Integration

$K = \int_V F(r,s,t)\, dr\, ds\, dt \quad \text{mit} \quad F(r,s,t) = B^T C B \, det(J)$

Newton-Cotes:

$$\int_a^b F(r)\, dr = (b-a) \sum_{i=0}^{n} C_i^n F_i + R_n$$

with C_i^n: Newton-Cotes constants; R_n: Remainder (error estimation)

The approximation points F_i are linearly distributed, for the distance between to points $h = \frac{b-a}{n}$ holds.

Example

The integral $\int_{-1}^{1} F(r)\, dr$ shall be approximated using

just one interval:

$$\int_{-1}^{1} F(r)\, dr = 2 \cdot \left(\tfrac{1}{2} F(-1) + \tfrac{1}{2} F(1) + R_1\right)$$

two intervals:

$$\int_{-1}^{1} F(r)\, dr = 2 \cdot \left(\tfrac{1}{6} F(-1) + \tfrac{4}{6} F(0) + \tfrac{1}{6} F(1) + R_2\right)$$

TABLE 5.5 *Newton-Cotes numbers and error estimates*

Number of intervals n	C_0^n	C_1^n	C_2^n	C_3^n	C_4^n	C_5^n	C_6^n	Upper bound on error R_n as a function of the derivative of F
1	$\frac{1}{2}$	$\frac{1}{2}$						$10^{-1}(b-a)^3 F^{II}(r)$
2	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$					$10^{-3}(b-a)^5 F^{IV}(r)$
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$				$10^{-3}(b-a)^5 F^{IV}(r)$
4	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$			$10^{-6}(b-a)^7 F^{VI}(r)$
5	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$		$10^{-6}(b-a)^7 F^{VI}(r)$
6	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$	$10^{-9}(b-a)^9 F^{VIII}(r)$

Centrale Differences Method

$^t\ddot{U} = \frac{1}{\Delta t^2}\left({}^{t-\Delta t}U - 2\,{}^tU +{}^{t+\Delta t} U\right) \qquad {}^t\dot{U} = \frac{1}{2\Delta t}\left(-{}^{t-\Delta t}U +{}^{t+\Delta t} U\right)$

Error is of order $(\Delta t)^2$.

System of equations at time t:

$M\,{}^tU + C\,{}^t\dot{U} + K\,{}^tU ={}^t R$

Because the system is concidered at time t, not at time t+Δt the method is called and explicit integration method.

Insert approximations for $^t\ddot{U}$ and $^t\dot{U}$:

$\left(\frac{1}{\Delta t^2}M + \frac{1}{2\Delta t}C\right){}^{t+\Delta t}U ={}^t R - \left(K - \frac{2}{\Delta t^2}M\right){}^tU - \left(\frac{1}{\Delta t^2}M - \frac{1}{2\Delta t}C\right){}^{t-\Delta t}U$

Since ^0U, $^0\dot{U}$ and $^0\ddot{U}$ are known, one only needs to calculate $^{-\Delta t}U$ to start the calculation of one time step after another.

From the equations for $^t\ddot{U}$ and $^t\dot{U}$ one obtains: $^{-\Delta t}U ={}^0 U - \Delta t^0\dot{U} + \frac{\Delta t^2}{2}{}^0\ddot{U}$

Now one can calculate $^{\Delta t}U$ and respectivly $^{t+\Delta t}U$ and then $^t\ddot{U}$ and $^t\dot{U}$ for every time step.

TABLE 9.1 *Step-by-step solution using central difference method (general mass and damping matrices)*

A. *Initial calculations:*

1. Form stiffness matrix $\mathbf{K}$, mass matrix $\mathbf{M}$, and damping matrix $\mathbf{C}$.
2. Initialize $^0\mathbf{U}$, $^0\dot{\mathbf{U}}$, and $^0\ddot{\mathbf{U}}$.
3. Select time step Δt, $\Delta t \le \Delta t_{cr}$, and calculate integration constants:

$$a_0 = \frac{1}{\Delta t^2}; \qquad a_1 = \frac{1}{2\,\Delta t}; \qquad a_2 = 2a_0; \qquad a_3 = \frac{1}{a_2}$$

4. Calculate $^{-\Delta t}\mathbf{U} = {}^0\mathbf{U} - \Delta t\,{}^0\dot{\mathbf{U}} + a_3\,{}^0\ddot{\mathbf{U}}$.
5. Form effective mass matrix $\hat{\mathbf{M}} = a_0\mathbf{M} + a_1\mathbf{C}$.
6. Triangularize $\hat{\mathbf{M}}$: $\hat{\mathbf{M}} = \mathbf{LDL}^T$.

B. *For each time step:*

1. Calculate effective loads at time t:

$$^t\hat{\mathbf{R}} = {}^t\mathbf{R} - (\mathbf{K} - a_2\mathbf{M})\,{}^t\mathbf{U} - (a_0\mathbf{M} - a_1\mathbf{C})\,{}^{t-\Delta t}\mathbf{U}$$

2. Solve for displacements at time $t + \Delta t$:

$$\mathbf{LDL}^T\,{}^{t+\Delta t}\mathbf{U} = {}^t\hat{\mathbf{R}}$$

3. If required, evaluate accelerations and velocities at time t:

$$^t\ddot{\mathbf{U}} = a_0({}^{t-\Delta t}\mathbf{U} - 2\,{}^t\mathbf{U} + {}^{t+\Delta t}\mathbf{U})$$
$$^t\dot{\mathbf{U}} = a_1(-{}^{t-\Delta t}\mathbf{U} + {}^{t+\Delta t}\mathbf{U})$$

Newmark Integration

Assumptions:

$$^{t+\Delta t}\dot{U} = ^{t}\dot{U} + \left[(1-\delta)\,^{t}\ddot{U} + \delta^{t+\Delta t}\ddot{U}\right]\Delta t$$

$$^{t+\Delta t}U = ^{t}U + ^{t}\dot{U}\Delta t + \left[\left(\tfrac{1}{2}-\alpha\right)^{t}\ddot{U} + \alpha^{t+\Delta t}\ddot{U}\right]\Delta t^2$$

Obtain $^{t+\Delta t}\ddot{U}$ from second equation, insert in first equation.

Insert the resulting eqations for $^{t+\Delta t}\dot{U}$ and $^{t+\Delta t}\ddot{U}$ in the euation of motion at time $t+\Delta t$.

$$M\,^{t+\Delta t}U + C\,^{t+\Delta t}\dot{U} + K\,^{t+\Delta t}U = ^{t+\Delta t}R$$

Because the system is concidered at time $t+\Delta t$, not at time t the method is called and implicit integration method.

TABLE 9.4 *Step-by-step solution using Newmark integration method*

A. *Initial calculations:*

1. Form stiffness matrix $\mathbf{K}$, mass matrix $\mathbf{M}$, and damping matrix $\mathbf{C}$.
2. Initialize $^0\mathbf{U}$, $^0\dot{\mathbf{U}}$, and $^0\ddot{\mathbf{U}}$.
3. Select time step Δt and parameters α and δ and calculate integration constants:

$$\delta \geq 0.50; \qquad \alpha \geq 0.25(0.5+\delta)^2$$

$$a_0 = \frac{1}{\alpha\,\Delta t^2}; \quad a_1 = \frac{\delta}{\alpha\,\Delta t}; \quad a_2 = \frac{1}{\alpha\,\Delta t}; \quad a_3 = \frac{1}{2\alpha} - 1;$$

$$a_4 = \frac{\delta}{\alpha} - 1; \quad a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right); \quad a_6 = \Delta t(1-\delta); \quad a_7 = \delta\,\Delta t$$

4. Form effective stiffness matrix $\hat{\mathbf{K}}$: $\hat{\mathbf{K}} = \mathbf{K} + a_0\mathbf{M} + a_1\mathbf{C}$.
5. Triangularize $\hat{\mathbf{K}}$: $\hat{\mathbf{K}} = \mathbf{LDL}^T$.

B. *For each time step:*

1. Calculate effective loads at time $t+\Delta t$:

$$^{t+\Delta t}\hat{\mathbf{R}} = ^{t+\Delta t}\mathbf{R} + \mathbf{M}(a_0\,^t\mathbf{U} + a_2\,^t\dot{\mathbf{U}} + a_3\,^t\ddot{\mathbf{U}}) + \mathbf{C}(a_1\,^t\mathbf{U} + a_4\,^t\dot{\mathbf{U}} + a_5\,^t\ddot{\mathbf{U}})$$

2. Solve for displacements at time $t+\Delta t$:

$$\mathbf{LDL}^T\,^{t+\Delta t}\mathbf{U} = ^{t+\Delta t}\hat{\mathbf{R}}$$

3. Calculate accelerations and velocities at time $t+\Delta t$:

$$^{t+\Delta t}\ddot{\mathbf{U}} = a_0(^{t+\Delta t}\mathbf{U} - ^t\mathbf{U}) - a_2\,^t\dot{\mathbf{U}} - a_3\,^t\ddot{\mathbf{U}}$$

$$^{t+\Delta t}\dot{\mathbf{U}} = ^t\dot{\mathbf{U}} + a_6\,^t\ddot{\mathbf{U}} + a_7\,^{t+\Delta t}\ddot{\mathbf{U}}$$

More stuff

More stuff

All figures and tables are taken from the book "Finite-Element-Procedures" by Klaus-Jürgen Bathe. It is the only source I used...

SoSe 11, Lars Radtke