A 100 € Problem

Notation: For $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$ denote by $|A|, |x|$ the matrix, vector of absolute values.
Hence $|A| \in \mathbb{R}^{n\times n}_{\geq 0}, |x| \in \mathbb{R}^n_{\geq 0}$.
For $x, y \in \mathbb{R}^n$ comparison is entrywise, i.e., $x \geq y \quad :\Leftrightarrow \quad x_i \geq y_i$ for $i \in \{1, ..., n\}$.
Denote $e \in \mathbb{R}^n$ with $e := (1, ..., 1)^T$.

Conjecture: $A \in \mathbb{R}^{n\times n}$ with $|A|e = ne \quad \Rightarrow \quad \exists\, 0 \neq x \in \mathbb{R}^n : |Ax| \geq |x|$.

The geometrical interpretation of the assumptions is that the rows of A lie on the octahedron centered at the origin with the nonzero entry of the vertices equal to n.

For a proof or counterexample of the conjecture I am happy to reward you with 100 €.

Note: Theorem 5.8 in [8] proved
$A \in \mathbb{R}^{n\times n}$ with $|A|e = ne \quad \Rightarrow \quad \exists\, 0 \neq x \in \mathbb{R}^n : |Ax| \geq \frac{1}{3+2\sqrt{2}}|x|$.

A stronger conjecture

Denote the i-th row of A by r_i. Dr. Florian Bünger proposed the following stronger formulation:

Conjecture 2: $A \in \mathbb{R}^{n\times n}, n \geq 2$ and $\|r_i\|_2 \geq \sqrt{n-1}$ for all $i \in \{1, \ldots, n\} \quad \Rightarrow$
$\exists\, 0 \neq x \in \mathbb{R}^n : |Ax| \geq |x|$.

This formulation is stronger because $|A|e = ne \;\Rightarrow\; n = |r_i|e \leq \|r_i\|_2\sqrt{n}$.

The geometrical interpretation of the assumptions of Conjecture 2 is that the rows of A lie on or outside the sphere centered at the origin with radius $\sqrt{n-1}$.

This second conjecture is particularly appealing because if true it is sharp in the following sense:

For any $\alpha > 1$ Conjecture 2 does not hold true when replacing the assertion by $|Ax| \geq \alpha|x|$.

A counterexample is the $n \times n$ matrix A with zero diagonal, all elements above the diagonal equal to 1 and all elements below the diagonal equal to -1.

The matrix satisfies the assumptions, and $|Ax| = |x|$ for $x = (1, 1, 0, \ldots)^T$.
However, it is shown in [8, Lemma 5.7] that there is no $x \in \mathbb{R}^n$ with $|Ax| > |x|$.

This sharpness of Conjecture 2 may offer more efficient proof schemes.

I am happy to reward you with 100 € for a proof or counterexample of Conjecture 2.
That implies that a proof of Conjecture 2 is worth 200 €.

Theoretical background

For $\mathbb{K} \in \{\mathbb{R}_{\geq 0}, \mathbb{R}, \mathbb{C}\}$ and a matrix $A \in \mathbb{K}^{n\times n}$ consider the quantity

$$\varrho^{\mathbb{K}}(A) := \max\{|\lambda| : \lambda \in \mathbb{K}, 0 \neq x \in \mathbb{K}^n\}. \tag{0.1}$$

For $\mathbb{K} = \mathbb{R}_{\geq 0}$ Perron-Frobenius Theory implies that this is the Perron root of a nonnegative matrix. For $\mathbb{K} = \mathbb{R}$ this quantity was introduced in [8] as the sign-real spectral radius, and for $\mathbb{K} = \mathbb{C}$ it was introduced in [11] and called the sign-complex spectral radius.

The sign-real spectral radius originated in the investigation of the componentwise distance to the nearest singular matrix [9, 7]. One reason why $\varrho^{\mathbb{K}}$ gained interest in matrix theory was that Collatz's [4] characterization of the Perron root extends to the sign-real and the sign-complex spectral radius:

$$A \in \mathbb{K}^{n\times n}: \qquad \varrho^{\mathbb{K}}(A) \; = \; \max_{\substack{x\in\mathbb{K}^n \\ x\neq 0}} \; \min_{x_i\neq 0} \left| \frac{(Ax)_i}{x_i} \right|. \tag{0.2}$$

Moreover, all three quantities $\varrho^{\mathbb{K}}$ share a number of other quantities with the Perron root:

$$\begin{array}{rcl} & & \varrho^{\mathbb{K}}(A^H) = \varrho^{\mathbb{K}}(A) \\ |S_1| = |S_2| = I & \Rightarrow & \varrho^{\mathbb{K}}(S_1 A S_2) = \varrho^{\mathbb{K}}(A) \\ P \in \mathbb{R}^{n\times n} \text{ permutation matrix} & \Rightarrow & \varrho^{\mathbb{K}}(P^T A P) = \varrho^{\mathbb{K}}(A) \\ D \in \mathbb{K}^{n\times n} \text{ nonsingular diagonal} & \Rightarrow & \varrho^{\mathbb{K}}(D^{-1} A D) = \varrho^{\mathbb{K}}(A) \\ \alpha \in \mathbb{K} & \Rightarrow & \varrho^{\mathbb{K}}(\alpha A) = |\alpha| \varrho^{\mathbb{K}}(A) \\ \mu \subseteq \{1, \ldots, n\} & \Rightarrow & \varrho^{\mathbb{K}}(A[\mu]) \leq \varrho^{\mathbb{K}}(A) \;\; \text{[inheritance]} \\ A \text{ triangular} & \Rightarrow & \varrho^{\mathbb{K}}(A) = \max_i |A_{ii}| \end{array}$$

The $\varrho^{\mathbb{K}}$ are continuous [8], and in [13] it was shown that the first four linear transformations and their combinations capture all linear operators leaving $\varrho^{\mathbb{R}}$ invariant. A number of other properties of $\varrho^{\mathbb{K}}$ are discussed in [3, 5, 6, 12] and the literature cited over there. Bounds using cycle products were investigated [10, 1], and in [2] a number of topological, invariance and other properties are discussed.

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