PAPER

On Nishi's conditions for Ω-property

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SUMMARY The concept of an Ω-matrix was introduced by Nishi in order to estimate the number of solutions of a resistive circuit containing active elements. He gave a finite characterization by means of four conditions which are all satisfied if and only if the matrix under investigation is an Ω-matrix. In this note we show that none of the four conditions can be omitted.

key words: *Ω-matrix, P-matrix, resistive circuit*

sectionIntroduction and result The concept of an Ω-matrix was introduced in connection with evaluation of the number of dc operating points of a transistor circuit. The definition is based on the so-called sign-condition.

Definition 1: A matrix $A \in M_n(\mathbb{R})$ is said to satisfy the sign-condition if, for each row of which the diagonal element is negative, all off-diagonal elements of that row are negative or zero.

Definition 2: A matrix $A \in M_n(\mathbb{R})$ is said to be an Ω-matrix or $A \in \Omega$, if $(A+D)^{-1}$ satisfies the sign-condition for all positive diagonal D whenever $A+D$ is nonsingular.

The concept of an Ω-matrix solves the original problem in the following way. The circuit equation of a transistor circuit can be written as

$$F(x) + Ax = b, \qquad (1)$$

where $A \in M_n(\mathbb{R}), x, b \in \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function with certain monotonicity properties. Then it was shown [4] that equation (1) has only finite many solutions if and only if $A \in \Omega$.

Besides solving the problem in circuit analysis, Ω-matrices have a number of remarkable properties as shown in [3], [4].

Theorem 1: Let A be an Ω-matrix and D be positive diagonal. Then

i) $A + D$ is an Ω-matrix,
ii) every principal submatrix of A is an Ω-matrix,
iii) if A is nonsingular, then A^{-1} is an Ω-matrix,
iv) AD and DA are Ω-matrices.

Furthermore

v) the class of P_0-matrices is a subset of Ω.

So Ω-matrices may be viewed as a generalization of P-matrices. It is, however, known that checking P- or P_0-property of a matrix is NP-hard [1]. And from the definition, the assertion of Ω-property of a matrix requires in principle infinitely many diagonal D to be tested. We note the similarity to P_0-matrices by the following theorem [2, Theorems 5.22 and 5.26].

Theorem 2: A real square matrix A is a P_0-matrix if and only if for each positive diagonal matrix D all minors of $A + D$ are nonzero.

Fortunately, the Ω-property is, compared to P-matrices, general enough to be checked in finite and even polynomial time. In [3], Nishi gave a finite test by showing that a real matrix A is an Ω-matrix if and only if four easy-to-verify conditions are satisfied. To formulate the theorem, we need some notation. The cofactor obtained by deleting rows $i_1, i_2, ..., i_k$ and columns $j_1, j_2, ..., j_k$ is denoted by

$$v\begin{pmatrix} i_1 & i_2 & ... & i_k \\ j_1 & j_2 & ... & j_k \end{pmatrix}$$

In particular, the determinant $|A|$ is denoted by v. Nishi proved the following theorem.

Theorem 3: A matrix A is an Ω-matrix if and only if each of the following conditions is satisfied:

i) A satisfies the sign condition.
ii) The inverse of each principal submatrix of A including A^{-1} satisfies the sign condition.
iii) If the conditions

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$$\frac{v\begin{pmatrix} i \\ i \end{pmatrix}}{v} > 0, \qquad \frac{v\begin{pmatrix} j \\ j \end{pmatrix}}{v} > 0$$

$$\text{and} \quad \frac{v\begin{pmatrix} i & j \\ i & j \end{pmatrix}}{v\begin{pmatrix} j \\ j \end{pmatrix}} < 0$$

are satisfied, then

$$\frac{v\begin{pmatrix} i \\ j \end{pmatrix}}{v} < 0.$$

iv) If the conditions

$$\frac{v\begin{pmatrix} i \\ i \end{pmatrix}}{v} < 0 \quad \textit{and} \quad \frac{v\begin{pmatrix} i & j \\ i & j \end{pmatrix}}{v\begin{pmatrix} j \\ j \end{pmatrix}} < 0$$

are satisfied, then

$$\frac{v\begin{pmatrix} j \\ j \end{pmatrix}}{v} > 0.$$

The computational effort is obviously not greater then $\mathcal{O}(n^5)$ because essentially computation of $n(n-1)/2$ minors suffices.

In his paper [3] Nishi posed the question whether one or more conditions in Theorem 3 can be left out. This would be the case if three of the four conditions in Theorem 3 would already imply the remaining one. We will show that this is not the case, i.e. that none of the four conditions can be omitted.

Theorem 4: For $i \in \{1,2,3,4\}$ there exists a matrix A_i such that A_i is not in Ω but satisfies all conditions $j \in \{1,2,3,4\}\backslash\{i\}$ of Theorem 3.

Proof. Define

$$A_1 = \begin{pmatrix} -12 & -4 & 8 \\ -22 & -6 & 14 \\ 15 & 5 & -9 \end{pmatrix}.$$

Then

$$(A_1 + 17I)^{-1} = 3654^{-1} \begin{pmatrix} -18 & -72 & 144 \\ -386 & 80 & 246 \\ 275 & 85 & 33 \end{pmatrix}$$

so $A_1 \notin \Omega$. The matrix A_1 does not satisfy the first, the sign condition, but the other three. It is

$$A_1^{-1} = 4^{-1} \begin{pmatrix} 4 & -1 & 2 \\ -3 & 3 & 2 \\ 5 & 0 & 4 \end{pmatrix}$$

and the inverses of the 2×2 principal submatrices are

$$16^{-1}\begin{pmatrix} 9 & 14 \\ 5 & 6 \end{pmatrix}, \; 12^{-1}\begin{pmatrix} 9 & 8 \\ 15 & 12 \end{pmatrix}$$

$$\text{and} \quad 8^{-1}\begin{pmatrix} 3 & -2 \\ -11 & 6 \end{pmatrix}.$$

Furthermore $v = \det A_1 = -16$, the 2×2 minors are $-16, -12, -16$ and the 1×1 minors are of course the diagonal elements. Therefore, conditions 3 and 4 are satisfied because their assumptions never apply.

Define

$$A_2 = 4A_1^{-1} = \begin{pmatrix} 4 & -1 & 2 \\ -3 & 3 & 2 \\ 5 & 0 & 4 \end{pmatrix}.$$

Then $A_2 \notin \Omega$ because $(A_2 + \varepsilon I)^{-1} = 4^{-1}A_1 + \mathcal{O}(\varepsilon)$ does not satisfy the sign condition for small enough ε. But A_2 itself satisfies the sign condition, and $v = -4$ and the 2×2 minors $12, 6, 9$ show that conditions 3 and 4 are satisfied because their assumptions do not apply.

Define

$$A_3 = \begin{pmatrix} 0 & 1 & -3 \\ 5 & 0 & -4 \\ -1 & -1 & 4 \end{pmatrix}.$$

Then

$$(A_3 + I)^{-1} = 8^{-1} \begin{pmatrix} -1 & 2 & 1 \\ 21 & -2 & 11 \\ 4 & 0 & 4 \end{pmatrix}$$

shows $A_3 \notin \Omega$. It is

$$A_3^{-1} = \begin{pmatrix} 4 & 1 & 4 \\ 16 & 3 & 15 \\ 5 & 1 & 5 \end{pmatrix}$$

and the inverses of the 2×2 principal submatrices are

$$4^{-1}\begin{pmatrix} -4 & -4 \\ -1 & 0 \end{pmatrix}, \; 3^{-1}\begin{pmatrix} -4 & -3 \\ -1 & 0 \end{pmatrix}$$

$$\text{and} \quad 5^{-1}\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}.$$

It follows that conditions 1 and 2 are satisfied. Furthermore, $v = -1$ and the 2×2 minors are $-4, -3, -5$, and $v\binom{k}{k}/v > 0$ for all $1 \le k \le 3$, so that condition 4 is satisfied. For $i = 1, j = 2$ it is $v\begin{pmatrix} i & j \\ i & j \end{pmatrix}/v\binom{i}{j} = 4/(-3) < 0$, so (6) is satisfied. But $v\binom{i}{j}/v = -\det\begin{pmatrix} 5 & -4 \\ -1 & 4 \end{pmatrix}/(-1) = 16 > 0$ contradicting (7), so that condition 3 is not satisfied.

Define

$$A_4 = \begin{pmatrix} -3 & 0 & 0 \\ 4 & 3 & 3 \\ 0 & -4 & -5 \end{pmatrix}.$$

Then

$$(A_4 + 4I)^{-1} = 5^{-1}\begin{pmatrix} 5 & 0 & 0 \\ 4 & -1 & -3 \\ -16 & 4 & 7 \end{pmatrix}$$

proving $A_4 \notin \Omega$. Furthermore,

$$A_4^{-1} = 9^{-1}\begin{pmatrix} -3 & 0 & 0 \\ 20 & 15 & 9 \\ -16 & -12 & -9 \end{pmatrix}$$

and the inverses of the 2×2 principal submatrices

$$3^{-1}\begin{pmatrix} 5 & 3 \\ -4 & -3 \end{pmatrix}, \; 15^{-1}\begin{pmatrix} -5 & 0 \\ 0 & -3 \end{pmatrix} \quad \text{and} \quad 9^{-1}\begin{pmatrix} -3 & 0 \\ 4 & 3 \end{pmatrix}$$

show that conditions 1 and 2 are satisfied. Furthermore, $v = 9$ and the 2×2 minors $-3, 15, -9$ show that $v\binom{i}{i}/v > 0$ is only satisfied for the one index $i = 2$, so condition 3 is valid because its assumptions are never satisfied. Finally, for $i = 1$ and $j = 3$ it is $v\binom{i}{i}/v = -3/9 < 0, v\begin{pmatrix} i & j \\ i & j \end{pmatrix}/v\binom{j}{j} = 3/(-9) < 0$ so that the assumptions (8) are satisfied. But $v\binom{j}{j}/v = (-9)/9 < 0$ contradicts (9), showing that condition 4 is not satisfied. $\square$

Remark. We note that no principal minor of A_i of size greater or equal to 2 is zero, so that all inverses necessary to validate condition 2 are well defined. We also note that we utilize that the sign condition is satisfied if, for negative diagonal element, off-diagonal elements are nonnegative. In each of the above examples the "boundary" of that condition is utilized by certain zero components.

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