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ON P-MATRICES

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Abstract. We present some necessary and sufficient conditions for a real matrix being P-matrix. They are based on the sign-real spectral radius and regularity of a certain interval matrix . We show that no minor can be left out when checking for P-property. Furthermore, a not necessarily exponential method for checking P-property is given.

1. Introduction and notation. A real matrix $A \in M_n(\mathbb{R})$ is called *P-matrix* if all its principal minors are positive. The class of P-matrices is denoted by $\mathcal{P}$. The *P-problem*, namely the problem of checking whether a given matrix is a P-matrix, is important in many applications, see [1].

A straightforward algorithm evaluating the $2^n - 1$ principal minors requires some $n^3 2^n$ operations. This corresponds to the fact that the P-problem is NP-hard [2]. In Theorem 2.2 we will show that none of these minors can be left out.

However, there are other strategies. Recently, Tsatsomeros and Li [20] presented an algorithm based on Schur complements reducing computational complexity to $7 \cdot 2^n$. The algorithm requires always this number of operations if the matrix in question is a P-matrix. Otherwise, the computational cost is frequently much smaller because one nonpositive minor suffices to prove $A \notin \mathcal{P}$.

In this paper we will present characterizations of P-matrices related to the sign-real spectral radius, and based on that some necessary conditions and sufficient conditions. In case $A \notin \mathcal{P}$ we also derive strategies to find a nonpositive minor. Finally, we give an algorithm which is not a priori exponential for $A \in \mathcal{P}$, but can be so in the worst case. The method is tested for $n = 100$, where all other known methods require 2^{100} operations. However, this approach needs further analysis.

We use popular notation in matrix theory. Especially, $A[\mu]$ denotes the principal submatrix of A with rows and columns out of $\mu \subseteq \{1, \ldots, n\}$. Absolute value and comparison of vectors and matrices is always to be understood componentwise. For example, signature matrices S are characterized by $|S| = I$.

2. Characterization of P-property. In [17] we introduced and investigated the sign-real spectral radius $\rho_0^{\mathfrak{S}}$. In the meantime we also introduced the sign-complex spectral radius. Therefore, for better readability, we changed the notation into $\rho^{\mathbb{R}}(A)$ for the sign-real and $\rho^{\mathbb{C}}(A)$ for the sign-complex spectral radius. The sign-real spectral radius is defined by

$$\rho^{\mathbb{R}}(A) := \max\{|\lambda| : \quad SAx = \lambda x,\ |S| = I,\ 0 \neq x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}\} \tag{1}$$

Note that the maximum is taken over the absolute values of *real* eigenvalues. Among the characterizations given in [17] is the following [Theorem 2.3]. For $0 < r \in \mathbb{R}$,

$$\rho^{\mathbb{R}}(A) < r \quad \Leftrightarrow \quad det(rI + SA) > 0 \text{ for all } |S| = I \tag{2}$$

$$\Leftrightarrow \quad det(rI + DA) > 0 \text{ for all } |D| \leq I. \tag{3}$$

This leads to two characterizations of the P-property.

THEOREM 2.1. *For $A \in M_n(\mathbb{R})$ and a positive r such that $\det(rI - A) \neq 0$ the following are equivalent:*

(*i*) $C := (rI - A)^{-1}(rI + A) \in \mathcal{P}$.
(*ii*) $\rho^{\mathbb{R}}(A) < r$.

For nonsingular A, parts (i) and (ii) are equivalent to
(*iii*) *All $B \in M_n(\mathbb{R})$ with $A^{-1} - r^{-1}I \leq B \leq A^{-1} + r^{-1}I$ are nonsingular.*

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Remark. The assertions follow by [17, Theorem 2.13 and Lemma 2.11]. Following, we give different and simpler proofs. This also allows to conclude the subsequent Theorem 2.2. As remarked by one referee, the assertions also follow by (2), (3) and [9, Theorem 3.4], see also [10, 18].

Proof. Let a fixed but arbitrary signature matrix S be given and define $\mu \subseteq \{1, \ldots, n\}$ by

$$\mu := \{i : \quad S_{ii} = 1\} \tag{4}$$

Define diagonal D by $D := \frac{1}{2}(I - S)$, so that $S = I - 2D$ and $D_{ii} = 0$ for $i \in \mu, D_{ii} = 1$ for $i \notin \mu$. Then $(I - D)C + D$ comprises of the rows of C out of μ, and the rows of the identity matrix out of $\{1, \ldots, n\} \setminus \mu$. Therefore,

$$\det((I - D)C + D) = \det C[\mu]. \tag{5}$$

On the other hand, $C = (rI - A)^{-1}(rI + A) = (rI + A)(rI - A)^{-1}$ and

$$\begin{aligned} (I - D)C + D &= \{(I - D)(rI + A) + D(rI - A)\}(rI - A)^{-1} \\ &= \{rI + A - 2DA\}(rI - A)^{-1} \\ &= (rI + SA)(rI - A)^{-1}, \end{aligned}$$

and in view of (5),

$$C \in \mathcal{P} \quad \Leftrightarrow \quad \forall |S| = I : \ \det(rI + SA)/\det(rI - A) > 0. \tag{6}$$

Now

$$\det(rI + SA) = \sum_{\omega} \det(SA)[\omega] \cdot r^{n - |\omega|},$$

where the sum is taken over all $\omega \subseteq \{1, \ldots, n\}$ including $\omega = \emptyset$. Summing the determinants over all S, all terms cancel except for $\omega = \emptyset$, such that

$$\sum_{|S|=I} \det(rI + SA) = 2^n \cdot r^n.$$

Therefore, not all $\det(rI + SA), |S| = I$, can be negative. This implies with (6),

$$C \in \mathcal{P} \quad \Leftrightarrow \quad \forall |S| = I : \ \det(rI + SA) > 0,$$

and proves $(i) \Leftrightarrow (ii)$. Concerning (iii), we use characterization (3) and a continuity argument to obtain

$$\begin{aligned} \rho^{\mathrm{IR}}(A) \geq r \quad &\Leftrightarrow \quad \exists\, |D| \leq I : \qquad \det(rI + DA) = 0 \\ &\Leftrightarrow \quad \exists\, |\widetilde{D}| \leq r^{-1} I : \quad \det(A^{-1} + \widetilde{D}) = 0. \end{aligned}$$

■

As a result of the previous proof we have a one-to-one correspondence between the minors of C and signature matrices S in (5) and (6): For $\det(rI - A) > 0$,

$$\det C[\mu] > 0 \quad \Leftrightarrow \quad \det(rI + SA) > 0 \tag{7}$$

for μ as defined in (4). As a result we obtain a solution to a question posed at our meeting in Oberwolfach.

THEOREM 2.2. *For every $n \geq 2$ and every $\emptyset \neq \mu \subseteq \{1, \ldots, n\}$, there exists a matrix $C \in M_n(\mathrm{IR})$ with*

$$\det C[\mu] < 0, \ \textit{and} \ \det C[\omega] > 0 \textit{ for all } \omega \subseteq \{1, \ldots, n\}, \ \omega \neq \mu.$$

Proof. Define $B := (1) \in M_n(\mathrm{IR})$, the matrix all components of which are 1's. Obviously, $\rho^{\mathrm{IR}}(B) = \rho(B) = n$. For every $|S| = I$, SB is of rank 1, so that the characteristic polynomial of SB is

$\chi_{SB}(x) = \det(xI - SB) = x^n - \mathrm{tr}(SB) \cdot x^{n-1}$. Therefore, $\chi_{SB}(x)$ is positive for $x > \max(0, \mathrm{tr}(SB))$. But $\mathrm{tr}(SB) \le n-2$ for all $|S| = I$, $S \ne I$, and $\mathrm{tr}(B) = n$. Hence, for every $n-2 < r < n$,

$$\det(rI - B) < 0, \text{ and } \det(rI - SB) > 0 \text{ for all } |S| = I, S \ne I. \tag{8}$$

Let $n \ge 2$ and $\emptyset \ne \mu \subseteq \{1, \ldots, n\}$ be given. Define $|S'| = I$ by

$$S'_{ii} = \begin{cases} 1 & \text{for } i \in \mu \\ -1 & \text{otherwise,} \end{cases}$$

and set $A := -S'B$. For fixed $r, n-2 < r < n$, define $C := (rI - A)^{-1}(rI + A)$. Then $S' \ne -I$ because $\mu \ne \emptyset$, and $\det(rI - A) > 0$ by (8). Furthermore, by (8),

$$\begin{aligned} S \ne S' \quad &\Rightarrow \quad \det(rI + SA) = \det(rI - SS'B) > 0, \\ S = S' \quad &\Rightarrow \quad \det(rI + SA) = \det(rI - B) < 0. \end{aligned}$$

Finally, the equivalence (7) finishes the proof. ■

The proof relies on the following fact. Let $A \in M_n(\mathrm{I\!R})$ and $r := \rho^{\mathrm{I\!R}}(A)$. Then there is $r' < r$ with $\det(r'I - \widetilde{S}A) < 0$ for some $|\widetilde{S}| = I$, and $\det(r'I - SA) > 0$ for all $|S| = I$, $S \ne \widetilde{S}$. This is explored in the proof for a specific matrix. We mention that, due to numerical experience, this seems by no means a rare case but rather typical for generic A and $r' \le r$, $r' \approx r$.

3. Necessary and sufficient conditions. In this section we present conditions for testing the P-property for a given matrix $C \in M_n(\mathrm{I\!R})$. First we make sure that the spectral radius of C is less than one. Set

$$\alpha = \|C\|_1 + 1; \quad \beta = 2^{\lceil \log_2 \alpha \rceil}; \quad C = C/\beta; \tag{9}$$

The P-property of C is not changed by the scaling; so we may assume without loss of generality that $I - C$ and $I + C$ are invertible.

We note that (9) is performed exactly (without rounding error) in IEEE 754 floating point arithmetic [6].

The inverse Cayley transform of $A := (C+I)^{-1}(C-I)$ is $C = (I-A)^{-1}(I+A)$. Note that since $\rho(C) < 1$, A is well defined. By Theorem 2.1 for $r = 1$, a lower bound on $\rho^{\mathrm{I\!R}}(A)$ yields a necessary condition for $C \in \mathcal{P}$, and an upper bound yields a sufficient condition for the P-property. This implies the following.

Theorem 3.1. *For $C \in M_n(\mathrm{I\!R})$ not having -1 as an eigenvalue define $A := (C+I)^{-1}(C-I)$. Then*

(i) $\quad C \in \mathcal{P} \quad \Rightarrow \quad \max\limits_{i,j} |A_{ij}A_{ji}|^{1/2} < 1.$

(ii) $\quad \|D^{-1}AD\|_2 < 1$ *for some diagonal* $D \quad \Rightarrow \quad C \in \mathcal{P}.$

Proof. Part (*i*) follows by $\max\limits_{i,j} |A_{ij}A_{ji}|^{1/2} \le \rho^{\mathrm{I\!R}}(A)$ [17, Lemma 5.1] and Theorem 2.1. Part (*ii*) follows for a maximizing S in (1) by

$$\rho^{\mathrm{I\!R}}(A) \le \rho(SA) = \rho(SD^{-1}AD) \le \|D^{-1}AD\|_2.$$ ■

The quantity

$$\inf_D \|D^{-1}AD\| \tag{10}$$

is a well known upper bound for the structured singular value [3]. It can be computed efficiently [22] using the fact that $\|e^{-D}Ae^{D}\|_2$ is a convex function in the D_{ii} [19].

Next we show that the sufficient condition (*ii*) in Theorem 3.1 is superior to certain other conditions for P-property.

THEOREM 3.2. *Let $C \in M_n(\mathrm{IR})$. Then*

(*i*) *$C + C^T$ positive definite implies that there exists $A := (C+I)^{-1}(C-I)$ and $\|A\|_2 < 1$.*

(*ii*) *C diagonally dominant with all diagonal elements positive implies that there exists $A := (C+I)^{-1}(C-I)$ and $\inf_D \|D^{-1}AD\|_2 < 1$, where the infimum is taken over all positive diagonal matrices.*

Proof. Part (*i*). If the Hermitian part of a matrix C is positive definite, then C has no nonpositive eigenvalues (follows by [13, Theorem 1] or by a field of values argument). Thus A is well defined. For $2(C + C^T) = (C + I)(C^T + I) - (C - I)(C^T - I)$ being positive definite, so is $I - (C + I)^{-1}(C - I)(C^T - I)(C^T + I)^{-1} = I - AA^T$.

Part (*ii*). Obviously, C has no nonpositive eigenvalues and thus A is well defined. The assumption implies that C is an H-matrix, so its comparison matrix is an M-matrix. By [5, Theorem 2.5.3.16] there exists a positive diagonal matrix D such that $CD^2 + D^2C^T$ is positive definite, and so is

$$I - D^{-1}(C + I)^{-1}(C - I)D^2(C^T - I)(C^T + I)^{-1}D^{-1}.$$

Therefore, $\rho((D^{-1}AD)(D^{-1}A^TD)) < 1$, and the assertion follows. ■

As we will see, minimizing over D in part (*ii*) of Theorem 3.1 provides frequently a fairly good sufficient condition for $C \in \mathcal{P}$. Weak cases exist, though practical experience suggests that they are rare (a class of such cases will be given in Section 3). At least, one can show that the ratio $\inf_D \|D^{-1}AD\|/\rho^{\mathrm{IR}}(A)$ is finite, though depending on n.

The necessary condition (*i*) in Theorem 3.1, however, can be arbitrarily weak. It may serve as an easy-to-compute first lower bound on $\rho^{\mathrm{IR}}(A)$.

Next we aim on a heuristic for computation of a lower bound on $\rho^{\mathrm{IR}}(A)$. Remember that every lower bound implies a necessary condition for $C \in \mathcal{P}$.

Define

$$\rho_0(A) := \max\{|\lambda| : \ \lambda \text{ real eigenvalue of } A\},$$

where $\rho_0(A) := 0$ if A has no real eigenvalue. For given $r > 0$ and $|S| = I$ with $\det(rI - SA) = 0$, the heuristic tries to alter S in order to increase the real eigenvalue r.

For given signature matrix S, suppose $r > 0$ is the largest real eigenvalue in absolute value of SA (in case $-r$ is an eigenvalue replace S by $-S$), so that $\det(rI - SA) = 0$ and $\det(r'I - SA) > 0$ for $r' > r$. A heuristic is to replace S by S' with

$$\det(rI - S'A) = \min\{\det(rI - \widetilde{S}A) : \ \widetilde{S} \text{ differs from } S \text{ in } m \text{ (diagonal) positions}\}. \tag{11}$$

The idea behind is that $\det(xI - SA) \to +\infty$ for $x \to +\infty$, so that $\det(rI - S'A) < 0$ implies existence of a real eigenvalue of $S'A$ greater than r. The smaller $\det(rI - S'A)$ is, that is the heuristic, the larger the new eigenvalue.

Obviously, the computational effort increases rapidly with m. Replacing S_{ii} by $-S_{ii}$ results in $S - 2S_{ii}e_ie_i^T$, and a computation using $\det(A + uv^T) = (1 + v^TA^{-1}u)\det A$ and $B := (r + \varepsilon I) - SA$ yields

$$\det(B + 2S_{ii}e_ie_i^TA) = \det B \cdot (1 + C_{ii}) \tag{12}$$

for $C := 2SAB^{-1}$. Similarly, for $i \neq j$,

$$\det(B + 2S_{ii}e_ie_i^T + 2S_{jj}e_je_j^T) = \det B \cdot ((1 + C_{ii})(1 + C_{jj}) - C_{ij}C_{ji}). \tag{13}$$

For C_{ii} denoting the minimal diagonal element of C, $S' = S - 2S_{ii}e_ie_i^T$ minimizes (11) for $m = 1$. Similarly, the minimal S' for $m = 2$ can be read off (13). Our heuristic is to determine the optimal S' for

$m = 2$, and then to calculate the maximum modulus of real eigenvalues of $S'A$. Therefore our heuristic merely identifies a new signature matrix S' with $\rho_0(S'A) > \rho_0(SA)$. Then, r is updated to $\rho_0(S'A)$. It may happen, by chance, that the largest real eigenvalue in absolute value of $S'A$ is negative. In this case S' is replaced by $-S'$.

This process is repeated until the minimum in (11) is positive, that is the maximum real eigenvalue r cannot be increased by our approach. The heuristic can be summarized in the following algorithm in pseudo-Matlab notation.

input: $A \in M_n(\mathbb{R})$
output: r with $r \le \rho^{\mathbb{R}}(A)$.
1) *Compute the real spectral radius* $r = \rho_0(A)$
2) *Make sure, A has a real eigenvalue*
$A(1,:) = -A(1,:);\ r1 = \rho_0(A);$
if $r1 > r,\ r = r1$; else $A(1,:) = -A(1,:)$; end
3) *Calculate determinantal correction for $i \neq j$*
$C = 2A(r(1+\varepsilon)I - A)^{-1};$
$d = 1 + \mathrm{diag}(C);\ E = d * d' - C .* C';$
4) *Take care of $i = j$*
$E = E - \mathrm{diag}(\mathrm{diag}(E)) + \mathrm{diag}(d);$
5) *Calculate minimum element E_{ij} and update*
$E_{ij} := \min\{E_{\mu\nu} :\ 1 \le \mu, \nu \le n\}$
if $E_{ij} > 0$, return, end
$A(i,:) = -A(i,:)$; if $i \neq j, A(j,:) = -A(j,:)$; end
$\lambda = \rho_0(A);$
6) *Make sure,* $\det(\lambda I - A) = 0$ if $\det(\lambda I - A) \neq 0, A = -A$; end
7) goto 3)

ALGORITHM 3.3. *Lower bound for $\rho^{\mathbb{R}}(A)$.*

Comment to step 2). For $S = \mathrm{diag}(-1, 1, \ldots, 1), \det(A) \le 0$ or $\det(SA) \le 0$, so that A or SA has a real eigenvalue because $\det(xI - A) \to +\infty$ for $x \to +\infty$.

Using this algorithm and (10) we obtain a necessary and a sufficient condition for $C \in \mathcal{P}$ as follows.

THEOREM 3.4. *Given $C \in M_n(\mathbb{R})$, set*

$$\alpha = \|C\|_1 + 1; \quad \beta = 2^{\lceil \log_2 \alpha \rceil}; \quad C = C/\beta;$$
$$A = (C + I)^{-1}(C - I);$$

Let lb denote the lower bound for $\rho^{\mathbb{R}}(A)$ computed by Algorithm 3.3, and let ub denote an upper bound $ub := \|D^{-1}AD\|_2$ for some positive diagonal D. Then

(i) necessary condition: $lb \ge 1 \quad \Rightarrow \quad C \notin \mathcal{P}$.
(ii) sufficient condition: $ub < 1 \quad \Rightarrow \quad C \in \mathcal{P}$.

The theorem is an immediate consequence of Theorem 2.1. Note that C is scaled such that $-1 \notin \lambda(C)$ and A is well defined. As has been mentioned before, $\inf \|D^{-1}AD\|_2$ can be efficiently approximated by convex programming, cf. [22].

The computing time for Algorithm 3.3 in its present form is about $3kn^3$, where k denotes the number of loops (steps 3-7). It can be reduced to about $k \cdot n^3$ by efficient calculation of the updated C in step 3.

The question remains, how sharp are the criteria in Theorem 3.4. We generated [1] three sets of matrices out

[1] Our special thanks to M. Tsatsomeros for pointing to these classes.

of $\mathcal{P}$:

$$(14)\qquad \begin{array}{rl} i) & C = B^{-1}A \text{ for row stochastic } A, B \text{ with positive diagonal.} \\ ii) & C = I + tA + t^2A^2 \text{ for } A > 0,\ \ \rho(A) < 1 \text{ and } 0 < t < 1. \\ iii) & C = B^{-1}A \text{ for } A, B \text{ upper Hessenberg, positive on and above,} \\ & \text{negative below the main diagonal.} \end{array}$$

For some random matrix $E \notin \mathcal{P}$ we define $M(t) = tE + (1-t)C$. Obviously, $M(0) \in \mathcal{P}$ and $M(1) \notin \mathcal{P}$. Finally, we adjusted the interval for t such that the crossing point from $\mathcal{P}$ to not $\mathcal{P}$ was approximately at $1/2$. We computed the maximum t_1 for which the sufficient criterion $\inf \|D^{-1}AD\|_2 < 1$ was still satisfied, and the minimum t_2 for which the necessary criterion $r < 1$ (r from Algorithm 3.3) was not satisfied, respectively. That means, $M(t) \in \mathcal{P}$ for $t \in [0, t_1]$, and $M(t) \notin \mathcal{P}$ for $t \in [t_2, 1]$. The values for t_1 and t_2 were calculated to relative precision 10^{-3}.

The following table lists the average and maximum ratio t_2/t_1 for the three test sets (14) and different dimensions, averaged over 10 samples each. In the two last columns, the average and maximum number k of loops in Algorithm 3.3 is listed for application to $M(t_2)$. Note that if Algorithm 3.3 is used specifically for the P-problem, it can stop when $r \geq 1$.

		t_2/t_1		k	
test set	n	average	maximum	average	maximum
i)	20	1.023	1.057	8.1	17
	50	1.069	1.292	20.2	34
	100	1.065	1.134	29.7	42
ii)	20	1.017	1.074	9.5	14
	50	1.054	1.101	17.4	21
	100	1.060	1.093	29.6	51
iii)	20	1.030	1.086	9.5	20
	50	1.042	1.129	14.3	20
	100	1.059	1.096	35.8	75

TABLE 3.5. *Test results averaged over 10 samples.*

The table shows that for these parametrized test sets the gap between the necessary condition and the sufficient condition given in Theorem 3.4 is not too large. This statement need not to extend to other test sets, as will be seen in the next section.

4. A not a priori exponential check of P-property.

Suppose for a given matrix $C \in M_n(\mathrm{IR})$ neither the necessary nor the sufficient condition of Theorem 3.4 is satisfied. In case $C \notin \mathcal{P}$, we may find some $\mu \subseteq \{1, \ldots, n\}$ with $\det C[\mu] \leq 0$ by some heuristic. However, in case $C \in \mathcal{P}$, and if no other criterion applies, the fastest known algorithm by Tsatsomeros and Li [20] requires some 2^n operations to verify $C \in \mathcal{P}$.

For general $C \in M_n(\mathrm{IR})$ there is not much hope to find an algorithm verifying $C \in \mathcal{P}$ in a computing time polynomially bounded in n, unless $P = NP$. However, this does not exclude that for specific C this is possible. And indeed, we will describe in the following an algorithm for checking P-property with not a priori exponential computing time in n, also for $C \in \mathcal{P}$. The worst case computing time, however, is exponential.

To be perfectly clear we are aiming on a so-called exact method for verifying P-property. The main property of such a method is that for each input matrix C it is decided in a finite number of steps whether $C \in \mathcal{P}$ or not. Certain heuristics are used to speed up this process; the decision, however, is rigorous.

In Theorem 2.1 (iii) we proved the P-property to be equivalent to nonsingularity of an interval matrix $\{\widetilde{A} \in M_n(\mathrm{IR}) :\ A^{-1} - r^{-1}I \leq \widetilde{A} \leq A^{-1} + r^{-1}I\}$, shortly written as $[A^{-1} - r^{-1}I, A^{-1} + r^{-1}I]$. Checking nonsingularity of an interval matrix is known to be NP-hard [16]. But Jansson gave in [7] an algorithm for calculating exact bounds for the solution set of a linear system where the matrix and the right hand vary within intervals. The most interesting and new property of this algorithm is that the computing time is not a priori exponential in the dimension n (although worst case). Based on that, an algorithm for checking regularity of an interval matrix with the same property concerning computing time was given in [8].

The basic idea is as follows. Given $[A] := \{\widetilde{A} \in M_n(\mathbb{R}) : \ \underline{A} \leq \widetilde{A} \leq \overline{A}\}$ for some $\underline{A}, \overline{A} \in M_n(\mathbb{R}), \underline{A} \leq \overline{A}$, and given $b \in \mathbb{R}^n$, define

$$\sum([A], b) := \{x \in \mathbb{R}^n : \quad \exists\, \widetilde{A} \in [A],\ \widetilde{A}x = b\}. \tag{15}$$

Then $\sum([A], b)$ is bounded iff $[A]$ is regular, i.e. iff $\forall\, \widetilde{A} \in [A] : \det \widetilde{A} \neq 0$. If $\sum$ is bounded, then it is connected; if $\sum$ is unbounded, then every (connected) component of $\sum$ is unbounded [7]. Therefore, the proof of regularity of $[A]$ is equivalent to check whether one component of $\sum$ is bounded or not.

It is well known that the smallest box parallel to the axes containing the intersection of $\sum$ with an orthant $\{Sx : x \geq 0\}$ of $\mathbb{R}^n$ for some $|S| = I$ can be characterized by a certain LP-problem [14]. The idea is now to solve $\widetilde{A}x = b$ for some $\widetilde{A} \in [A]$, and to start with the orthant x belongs to. If $\sum$ is unbounded in that orthant, $[A]$ is singular. If not, all neighboring orthants $\{S'x : \ x \geq 0\}$, where S' and S differ in exactly one entry, are checked. This process is continued until either $\sum$ is found to be unbounded in some orthant or, all neighboring orthants have empty intersection with $\sum$. In the first case $[A]$ is singular, in the latter $[A]$ is regular.

Clearly the computational effort is proportional to the number of orthants with nonempty intersection with $\sum$, and this number depends for given $[A]$ especially on the right hand side b. In [8] the authors give some heuristic how to choose b (dependent on $[A]$) to keep this number small.

In our special application we use the following theorem.

Theorem 4.1. *Let $C \in M_n(\mathbb{R})$ be given and assume* $\det(I - C) \cdot \det(I + C) \neq 0$. *Define* $A := (C - I)^{-1}(C + I)$ *and* $[A] := \{\widetilde{A} : \ A - I \leq \widetilde{A} \leq A + I\}$. *Then the following are equivalent:*

(i) $C \in \mathcal{P}$.
(ii) $[A]$ *is nonsingular.*

Furthermore, for every signature matrix S and every $b \in \mathbb{R}^n$,

$$\sum([A], b) \cap \{Sx : \ x \geq 0\} = \{x \geq 0 : \ (AS - I) \cdot x \leq b,\ (-AS - I) \cdot x \leq -b\}. \tag{16}$$

The *proof* follows by Theorem 2.1 and [7, Section 3], see also the remark after Theorem 2.1.

Following our previous remarks we are only interested in whether the feasible set of the right hand side in (16) is empty or not, that is we only need to execute Phase I of the simplex method. Thus we use the trivial objective function $f(x) = 0$ were every feasible point is optimal.

With these preliminaries we can formulate an algorithm for checking P-property. For $x \in \mathbb{R}^n$, define $s := \text{signum}(x) \in \mathbb{R}^n$ with $s_i := 1$ for $x_i \geq 0$, $s_i := -1$ otherwise. The neighborhood $N(s)$ is defined by $N(s) := \{(s_1, \ldots, s_{i-1}, -s_i, s_{i+1}, \ldots, s_n)^T : \ 1 \leq i \leq n\}$.

input: $C \in M_n(\mathbb{IR})$
output: $is\_P = 1$ if $C \in \mathcal{P}$, $is\_P = 0$ if $C \notin \mathcal{P}$

1) *Make sure* $\det(I - C) \cdot \det(I + C) \neq 0$, *and compute* A
   $\alpha = \|C\|_1 + 1$; $\beta = 2^{\lceil \log_2 \alpha \rceil}$; $C = C/\beta$;
   $A = (C - I)^{-1}(C + I)$;
2) *Choose right hand side* b
3) *Compute start orthant and initialize*
   $x = A^{-1}b$; $s = \mathrm{signum}(x)$; $L := \{s\}$; $T = \emptyset$;
4) *Check orthants*
   choose $s \in L$; $S = \mathrm{diag}(s)$; $L = L \setminus \{s\}$; $T = T \cup \{s\}$;
   set $\Omega := \{x \geq 0 : (AS - I) \leq b, \ (-AS - I) \leq -b\}$;
   if Ω *is unbounded* then $is\_P = 0$; return; end
   if $\Omega \neq \emptyset$ then $L = L \cup \{N(s) \setminus (L \cup T)\}$; end
5) if $L = \emptyset$ then $is\_P = 1$; return;
   else goto 4); end

Algorithm 4.2. *Checking P-property*

For the choice of the right hand side we use the same heuristic as in [8, Section 7]. The computational effort for Phase I of the simplex algorithm is $0(n^3)$, so the total computing time for Algorithm 4.2 is $0(k \cdot n^3)$, where k is the number of orthants checked, i.e. the length of the list T after execution.

A practical check for P-property combines our methods to a hybrid algorithm. First, the necessary and sufficient conditions from Theorem 3.4 are checked. If they fail and n is small, the algorithm by Tsatsomeros and Li is applied. If n is large, Algorithm 4.2 is used.

Following, we construct a set of parametrized matrices for which we know the exact value of the parameter where the P-property is lost and, for which neither the necessary nor the sufficient criterion of Theorem 3.4 is satisfied for a wide range of the parameter.

Consider

$$A = A_n := \begin{pmatrix} 0 & & +1 \\ & \ddots & \\ -1 & & 0 \end{pmatrix} \in M_n(\mathbb{IR}), \tag{17}$$

a skew-symmetric matrix with entries $+1$ above and -1 below the main diagonal. In [17, Lemma 5.6] we proved $\rho^{\mathbb{IR}}(A) = 1$ for every $n \geq 2$ by exploring characterization (2). A simpler proof uses that $|(I - A)^{-1}(I + A)|$ is a permutation matrix. By Theorem 2.1, $C = C(r) = (rI - A)^{-1}(rI + A) \in \mathcal{P}$ for every $r' > 1$. Moreover, an upper bound $\|D^{-1}AD\|_2$ is not only an upper bound for the real eigenvalues of all SA, $|S| = I$, but also for the complex eigenvalues. Especially, $\rho(A) \leq \|D^{-1}AD\|_2$ for every positive diagonal D. One can show that $\rho(A_n) = \sin(\pi/n)/(1 - \cos(\pi/n))$ with the limit $2n/\pi$. It follows that for all $n \geq 2$ and $1 < r < \rho(A_n)$,

- $C := (rI - A)^{-1}(rI + A) \in \mathcal{P}$, and
- neither of the criteria in Theorem 3.4 is satisfied.

As an example, $\rho(A_{20}) = 12.7$, $\rho(A_{50}) = 31.8$, $\rho(A_{100}) = 63.7$. That is for $1 < r < 63$ we cannot verify by our criteria so far that $(rI - A_{100})^{-1}(rI + A_{100}) \in \mathcal{P}$, and every known algorithm would require some $0(2^{100})$ operations.

We tested Algorithm 4.2 for $n \in \{20, 50, 100\}$ and several values of r. Note that for $r \geq 14, 32, 64$ for $n = 20, 50, 100$, respectively, $C \in \mathcal{P}$ by Theorem 3.4 (*ii*). The results are listed in Table 4.3, where from left to right we list r, the number *north* of orthants with nonempty intersection with $\sum([A], b)$, and the number *northchkd* of orthants checked. The total computational effort is $0(northchkd \cdot n^3)$. Some $***$ denote that the algorithm stopped without result due to memory limitations.

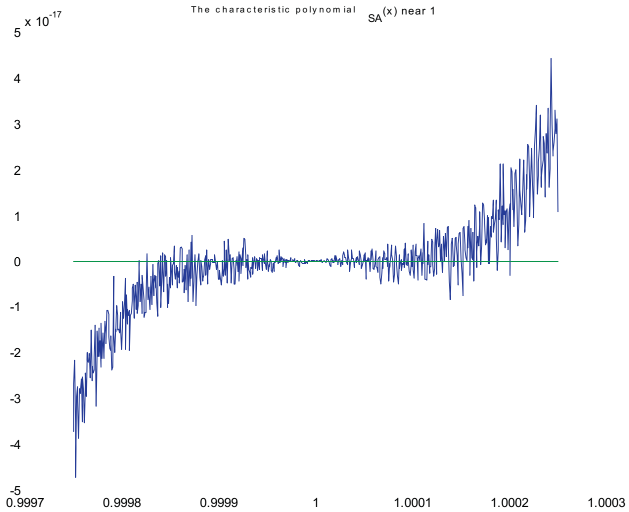
	$n = 100$		$n = 50$		$n = 20$	
r	north	northchkd	north	northchkd	north	northchkd
62	66	6419				
60	40	3924				
58	42	4113				
56	49	4780				
54	98	9467				
52	85	8247				
50	79	7671				
48	72	7001				
46	69	6713				
44	70	6811				
42	73	7099				
40	77	7483				
38	82	7963				
36	305	28903				
34	110	10570				
32	43	4215				
30	64	6229	26	1252		
28	65	6309	31	1485		
26	450	42454	39	1845		
24	59	5747	62	2889		
22	496	46483	95	4379		
20			109	5007		
18	317	29920	160	7260		
16	***		29	1396		
14	36	3529	39	1855		
12	***		78	3629	18	321
10	***		539	23600	37	616
8	54	5251	55	2593	70	1112
6	***		***		77	1207
4	29	2846	27	1300	19	346
2	***		***		651	7999

TABLE 4.3. *Results of Algorithm 4.2.*

Before interpreting the results, we discuss some numerical issues. We used the NAG library [15], algorithm E04MBF for linear programming. Occasionally, this algorithm stopped with error code IFAIL=4, which means that the limit on the number of iterations has been reached. For the objective function being constant zero this means that no feasible point has been found, yet. We ran extensive tests increasing the maximum number of iterations by a factor 10000, and either obtained a message "no feasible point found" or, still the same error code. Therefore, we interpreted this error code as the problem being not feasible.

Furthermore, the matrices $rI - SA$ for r near 1, A as in (17), may become very ill-conditioned for certain signature matrices S. Consider n even and $S := \mathrm{diag}(1, -1, \dots, -1, 1, \dots, 1)$ with $n/2$ entries -1. One can show that $\det(xI - SA) = (x^2 - 1)^{n/2}$, with one Jordan block of SA of size $n/2$ corresponding to the eigenvalues $+1$ and -1, respectively. Thus the sensitivity of the eigenvalues is $\varepsilon^{2/n}$ [4], where ε denotes the relative rounding error unit. Thus it is numerically difficult to calculate the sign of $\det(rI - SA)$ for r near 1. In the following graph, $\det(rI - SA)$ is drawn against r for $n = 10$ and $0.9997 \le r \le 1.0003$, computed in double precision IEEE 754 [6], corresponding to a precision of 16 decimal places.

For $0.9998 \lesssim r \lesssim 1.0002$ the sign cannot be decided. A multiple precision calculation using Maple [21]



Graph 4.4. *The characteristic polymomial of SA near 1.*

computed $\mathrm{cond}(1.0002I - SA) \sim 5 \cdot 10^{20}$. Correspondingly the numerical computation of the eigenvalues of SA suffers from the ill-conditioning. For example, Matlab [11] computes 1.0013 to be a (real) eigenvalue of SA for $n = 10$ (sic!), where we know that $\rho^{\mathbb{R}}(A) = 1$.

Needless to say that for higher dimensions things get much worse. Therefore (numerically) it makes not much sense to choose values r too close to 1 in Table 4.3.

The preceeding discussion is meant as a disclaimer to the results displayed in Table 4.3. The results may, at least partially, be numerical artefacts. Besides that, some 10^5 checked orthants for $n = 100$ corresponding to $10^5 n^3 = 10^{11}$ operations is not too much compared to 2^{100}.

Finally we mention that we tried to apply Algorithm 4.2 to the samples in Table 3.5 for $t_1 < t < t_2$. Unfortunately, the results were very poor. For $n = 20$, between 10^3 and 10^4 orthants had to be checked corresponding to 10^7 and 10^8 operations. Here the algorithm of Tsatsomeros and Li is better. For $n = 50$, Algorithm 4.2 regularly ran out of memory. The reason may be that the parameter t is already fairly close to the critical value where P-property is lost.

The worst case computing time of Algorithm 4.2 is exponential in n and, unless $P = NP$, an algorithm for finding a right hand side b such that the number of orthants with nonempty intersection with $\sum([A], b)$ is polynomially bounded in n is also exponential - if such a right hand side exists at all in general. We mention that in [12] a 3×3 example is given were $\sum([A], b)$ is not contained in one orthant for every right hand side b.

The results in Table 4.3 look promising: frequently not too many of the 2^n orthants intersect with $\sum$. Is this due to the specific example? Are there better heuristics to keep this number of orthants small?

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