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ILL-CONDITIONED MATRICES ARE COMPONENTWISE NEAR TO SINGULARITY

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Abstract. For a square matrix normed to 1, the normwise distance to singularity is well known to be equal to the reciprocal of the condition number. In this paper we give an elementary and self-contained proof for the fact that an ill-conditioned matrix is also not far from a singular matrix in a componentwise sense. This is shown to be true for any weighting of the componentwise distance. In words: Ill-conditioned means for matrix inversion nearly ill-posed also in the componentwise sense.

1. Introduction. The classical normwise condition number measures the sensitivity of the inverse of a matrix in a normwise sense. Given a real $n \times n$ matrix A, which we always assume to be nonsingular, and a matrix norm $||\cdot||$, the condition number may be defined by

$$\kappa(A) := \lim_{\epsilon \to 0^+} \sup_{||\delta A|| \le \epsilon \cdot ||A||} \frac{||(A+\delta A)^{-1} - A^{-1}||}{\epsilon ||A^{-1}||}.$$

It is well known that this condition number can be characterized by

$$\kappa(A) = ||A^{-1}|| \, ||A||.$$

For matrix norms subordinate to a vector norm and the matrix A normed to 1, it is also well known that the reciprocal of the condition number is equal to the normwise distance to the nearest singular matrix

$$\min\{0 \le \alpha \in \mathbf{R} \mid \exists \tilde{A} : ||\tilde{A} - A|| \le \alpha \cdot ||A|| \text{ and } \tilde{A} \text{ singular}\} = \frac{1}{\kappa(A)}$$

(see, for example, [14, Theorem 2.8]). Thus, an ill-conditioned matrix is nearby a singular matrix in the normwise sense.

Here and in the following of the paper, absolute value and comparison of matrices are always to be understood *componentwise*, for example $|\tilde{E}| \le \alpha E \;\; :\Leftrightarrow \;\; |\tilde{E}_{ij}| \le \alpha E_{ij}$ for $1 \le i, j \le n$.

Componentwise perturbation analysis for matrix inversion or the solution of linear systems leads to the Bauer-Skeel condition number ([2], [13])

$$\text{cond}_{BS}(A, E) = || \, |A^{-1}| \, E \, ||, \tag{1}$$

where E is some nonnegative $n \times n$ weight matrix. Note that (1) defines a condition number with respect tso the distance induced by E. The apparent contradiction, that replacing E by $2E$ doubles $\text{cond}_{BS}(A, E)$ resolves because the distance has been changed. Usually, A and E should have norms of the same size. For the important case of componentwise relative perturbations, $E = |A|$, this is naturally the case.

The condition number (1) is not independent of diagonal column scaling. The optimal Bauer-Skeel condition (see [3]) has been defined by

$$\text{cond}(A, E) := \inf_D \text{cond}_{BS}(AD, ED).$$

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This condition number is independent of diagonal row and column scaling. The same remark as above applies: The condition number is defined with respect to the distance induced by E.

Denoting the spectral radius by ρ, it is for any p-norm (see [1])

$$(2)\qquad \mathrm{cond}(A,E) = \rho(|A^{-1}|E) \quad \text{and} \quad \mathrm{cond}(A,|A|) = \inf_{D_1,D_2} \mathrm{cond}_\infty(D_1AD_2).$$

That means for suitably scaled matrices and relative perturbations, the ∞-norm condition number and the (componentwise) Bauer-Skeel condition number are identical, and the reciprocal of this number is the ∞-norm distance to the nearest singular matrix. Similar to the normwise case, it seems natural to ask whether a matrix being ill-conditioned in the componentwise sense is not too far from a singular matrix in the same componentwise sense.

Define the componentwise distance to the nearest singular matrix weighted by a nonnegative matrix E by

$$(3)\qquad \sigma(A,E) := \min\{\, \alpha \geq 0 \mid \exists \tilde{E} \text{ with } |\tilde{E}| \leq \alpha E \text{ and } A+\tilde{E} \text{ singular}\}.$$

If no such α exists, we set $\sigma(A,E) := \infty$. The set $\{\det \tilde{A} \mid |\tilde{A}-A| \leq \sigma(A,E)\cdot E\}$ and a compactness argument show that the minimum instead of an infimum may be used in (3).

The question above was, is the reciprocal $\mathrm{cond}(A,E)^{-1}$ of the componentwise condition number not too far from the componentwise distance $\sigma(A,E)$ to the nearest singular matrix?

This has been conjectured for the important case of relative perturbations $E = |A|$ by Demmel and N. Higham [3], and Higham writes [5]: "*This conjecture is both plausible and aesthetically pleasing because* $\sigma(A,|A|)$ *is invariant under two-sided diagonal scalings of* A *and* $\rho(|A^{-1}|\cdot|A|)$ *is the minimum* ∞*-norm condition number achievable by such scalings.*"

We solve this conjecture in the affirmative for arbitrary weight matrices E by proving

$$(4)\qquad \frac{1}{\rho(|A^{-1}|E)} \;\leq\; \sigma(A,E) \;<\; \frac{(3+2\sqrt{2})n}{\rho(|A^{-1}|E)}.$$

We note that we cannot hope to find a computationally simple formula or even algorithm for $\sigma(A,E)$ since Poljak and Rohn [7] showed that computation of $\sigma(A,E)$ is NP-hard. We also note that the lower bound in (4) is well known and sharp, and the upper bound in (4) is almost sharp in the sense that the factor $(3+2\sqrt{2})$ cannot be replaced by 1. The latter is because there are general $n \times n$ examples with $\sigma(A,|A|) = n/\rho(|A^{-1}||A|)$.

The result (4) has been proven in [9], and it uses extensively results obtained in [10] and especially [11]. The latter paper develops a Perron-Frobenius theory for matrices without sign restrictions. In the following, a self-contained, simplified and elementary proof of (4) is presented using only basic facts from linear algebra.

The paper is organized as follows. In the second chapter we give some characterizations of the componentwise distance to the nearest singular matrix. In those results, the sign-real spectral radius ρ_0^S, which is characterized in Chapter 3, plays a key role. It turns out that lower bounds on ρ_0^S imply upper bounds for $\sigma(A,E)$, the hard part in (4). Such bounds are derived in Chapter 4. Finally, the pieces are put together in Chapter 5 to prove our main result (4).

2. Characterizations of $\sigma(A,E)$. Define the *real spectral radius* [8] by

$$\rho_0(A) := \max\{|\lambda| \mid \lambda \text{ is } \textit{real} \text{ eigenvalue of } A\}.$$

We set $\rho_0(A) := 0$ if A has no real eigenvalue. Let a nonsingular real $n \times n$ matrix A and a nonnegative $n \times n$ matrix E be given, and denote the $n \times n$ identity matrix by I. Recall that absolute value and comparison will always be used componentwise.

For $|\tilde{E}| \le E$ and $|\lambda| = \rho_0(A^{-1}\tilde{E}) \ne 0$, the matrix $\lambda I - A^{-1}\tilde{E}$ is singular. Multiplying by $\lambda^{-1}A$ yields that $A - \lambda^{-1}\tilde{E}$ is singular with $|\lambda^{-1}\tilde{E}| \le \lambda^{-1}E$. Therefore, $\sigma(A,E) \le \lambda^{-1}$ and

$$\sigma(A,E) \le \frac{1}{\max\limits_{|\tilde{E}|\le E} \rho_0(A^{-1}\tilde{E})} \quad . \tag{5}$$

Conversely, suppose $A + \tilde{F}$ is singular for $|\tilde{F}| \le \alpha E$ and $\alpha := \sigma(A,E)$. Recall that A is always assumed to be nonsingular and therefore $\alpha > 0$. Then $\alpha^{-1}I + A^{-1} \cdot \alpha^{-1}\tilde{F}$ is singular as well, and therefore $\rho_0(A^{-1} \cdot \alpha^{-1}\tilde{F}) \ge \alpha^{-1}$. With $\tilde{E} := \alpha^{-1}\tilde{F}$ it is $\sigma(A,E) = \alpha \ge \{\rho_0(A^{-1}\tilde{E})\}^{-1}$ and $|\tilde{E}| \le E$. Together with (5) this proves the following characterization for the componentwise distance to the nearest singular matrix $\sigma(A,E)$ as defined in (3):

$$\sigma(A,E) = \frac{1}{\max\limits_{|\tilde{E}|\le E} \rho_0(A^{-1}\tilde{E})} \, . \tag{6}$$

This includes the case $\infty = 1/0$. In (6) the maximum is taken over infinitely many matrices $\tilde{E}$. We mention (but do not need) that it is not difficult to see that the maximum in (6) can be restricted to $|\tilde{E}| = E$. This is because $\det(\lambda I - A^{-1}\tilde{E})$ depends linearly on perturbations of a single entry of $\tilde{E}$, and for $\rho_0(A^{-1}F) = \max\limits_{|\tilde{E}|\le E} \rho_0(A^{-1}\tilde{E}) = \lambda$ and $|F_{ij}| < E_{ij}$ one can conclude from the definition of ρ_0 that $\det(\lambda I - A^{-1}F) = 0$ independent of F_{ij}, so especially for $F_{ij} := E_{ij}$.

Now we can prove the lower bound for $\sigma(A,E)$ in (4). For each $|\tilde{E}| \le E$,

$$\rho_0(A^{-1}\tilde{E}) \le \rho(A^{-1}\tilde{E}) \le \rho(|A^{-1}\tilde{E}|) \le \rho(|A^{-1}| \cdot E),$$

which proves the left inequality in (4).

The real spectral radius need not be continuous in the matrix components as $A(\epsilon) = \begin{pmatrix} 1 & \epsilon \\ -1 & 1 \end{pmatrix}$ shows with $\rho_0(A(\epsilon)) = 0$ for $\epsilon > 0$, and $\rho_0(A(0)) = 1$. As a matter of fact, $\sigma(A,E)$ depends continuously on A and E [10, Lemma 6.1]. That means the maximum in (6) must introduce some smoothing. We will identify this smoothing in order to obtain a continuous quantity characterizing $\sigma(A,E)$.

Define a signature matrix S to be diagonal with $S_{ii} \in \{+1,-1\}$, that means $|S| = I$. Then for any such S,

$$\begin{aligned} \sigma(A,E)^{-1} &= \max_{|\tilde{E}|\le E} \rho_0(A^{-1}\tilde{E}) \\ &= \max_{|\tilde{E}|\le E} \rho_0(SA^{-1}\tilde{E}S) \\ &= \max_{|\tilde{E}|\le E} \rho_0(SA^{-1}\tilde{E}). \end{aligned} \tag{7}$$

Since this equality holds for any signature matrix S, we define the *sign-real spectral radius* by

$$\rho_0^S(A) := \max_{\tilde{S}} \rho_0(\tilde{S}A), \tag{8}$$

where the maximum is taken over all signature matrices $\tilde{S}$. Then by (7) we may replace ρ_0 by ρ_0^S in (6) and obtain the characterization

$$\sigma(A,E) = \frac{1}{\max\limits_{|\tilde{E}|\le E} \rho_0^S(A^{-1}\tilde{E})} \quad . \tag{9}$$

We note that Rohn [8] showed $\sigma(A,E)^{-1} = \max_{S_1,S_2} \rho_0(S_1A^{-1}S_2E)$. For the purpose of proving (4) we only need (9).

3. Characterizations of ρ_0^S. The sign-real spectral radius has been defined and thoroughly investigated in [11]. In the following we list only those properties of ρ_0^S (with new proofs) which are necessary to prove the right inequality in (4).

For nonsingular diagonal D, the set of eigenvalues of A and $D^{-1}AD$ are identical, and signature matrices are idempotent. Hence, for signature matrices S_1, S_2,

$$\rho_0^S(A) = \rho_0^S(D^{-1}AD) = \rho_0^S(S_1AS_2). \tag{10}$$

Moreover,

$$\rho_0^S(\alpha A) = |\alpha| \cdot \rho_0^S(A) \quad \text{for } \alpha \in \mathbf{R}. \tag{11}$$

Using the characterization (9) yields for nonsingular diagonal D_1, D_2, and nonzero α,

$$\begin{aligned} &\sigma(D_1AD_2, |D_1ED_2|) = \sigma(A,E) \quad \text{and} \\ &\sigma(\alpha A, E) = \sigma(A, |\alpha^{-1}|E) = |\alpha| \cdot \sigma(A,E). \end{aligned} \tag{12}$$

Next we derive a simple-to-compute lower bound for ρ_0^S. This will be one of the keys to prove the right inequality in (4). It will also yield a characterization of ρ_0^S identifying it as the extension of the Perron-Frobenius theory to real matrices without sign restriction.

We will use the linearity of the determinant subject to rank-1 updates. Although well known, this result does not seem to belong to the repertoire of standard text books. For an $n \times n$ matrix A and $u, v \in \mathbf{R}^n$,

$$\det(A + \alpha uv^T) = \det A + \alpha \cdot v^T \operatorname{adj} A \cdot u, \tag{13}$$

where *adj* denotes the classical adjoint. To prove (13) for nonsingular A use

$$\begin{aligned} \det(A+\alpha uv^T) &= \det A \cdot \det(I + \alpha A^{-1}uv^T) = \det A \cdot \prod \lambda_i(I + \alpha A^{-1}uv^T) \\ &= \det A \cdot \prod (1 + \alpha\lambda_i(A^{-1}uv^T)) = \det A \cdot (1 + \alpha v^T A^{-1} u) \end{aligned} \tag{14}$$

denoting eigenvalues by λ_i and using the fact that the set of nonzero eigenvalues of BC and CB is identical for $B \in M_{m,n}(\mathbf{R})$, $C \in M_{n,m}(\mathbf{R})$. For singular A use a continuity argument.

THEOREM 3.1. *Suppose A is a real $n \times n$ matrix, $0 \ne x \in \mathbf{R}^n$ and $0 \le r \in \mathbf{R}$. Then*

$$|Ax| \ge r \cdot |x| \quad \Rightarrow \quad \rho_0^S(A) \ge r.$$

Proof. Define diagonal D by

$$D_{ii} := \begin{cases} r|x_i|/|(Ax)_i| & \text{for } (Ax)_i \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $(Ax)_i = 0$, then $|(Ax)_i| \geq r|x_i|$ implies $rx_i = 0$. Therefore, $D \cdot |Ax| = r \cdot |x|$ with $0 \leq D_{ii} \leq 1$. There are signature matrices S_1, S_2 with $S_1 Ax = |Ax|$ and $x = S_2|x|$, and for $\hat{D} := S_2 D S_1$ it follows $\hat{D}Ax = rx$. Now $x \neq 0$ yields

$$\det(rI - \hat{D}A) = 0 \quad \text{with} \quad -1 \leq \hat{D}_{ii} \leq 1. \tag{15}$$

We construct a signature matrix S with $\det(rI - SA) \leq 0$. For fixed index i, $1 \leq i \leq n$, define $\tilde{D} = \tilde{D}(\alpha)$ by

$$\tilde{D}_{\nu\nu} := \begin{cases} \hat{D}_{\nu\nu} & \text{for} \quad \nu \neq i \\ \alpha & \text{for} \quad \nu = i. \end{cases}$$

Then the difference between $rI - \hat{D}A$ and $rI - \tilde{D}A$ is of rank 1, and using (13), (15) and especially $-1 \leq \hat{D}_{ii} \leq 1$ it follows that $\det(rI - \tilde{D}A) \leq 0$ for $\alpha = 1$ or $\alpha = -1$. Repeating this argument for all indices i, $1 \leq i \leq n$, we obtain a signature matrix S with

$$\det(rI - SA) \leq 0.$$

This is the value of the characteristic polynomial $P(t) = \det(tI - SA)$ of SA at the nonnegative point $t = r$. Because $P(t) \to +\infty$ for $t \to +\infty$, $P(t)$ must cross the real axis for some $t^* \geq r$. Now (10) implies

$$r \leq t^* \leq \rho_0(SA) \leq \rho_0^S(A).$$

□

The last argument in the proof of Theorem 3.1 also implies that SA must have a real eigenvalue λ for some $S \in \mathcal{S}$ (recall that A need not have a real eigenvalue if n is even) because for at least half of all 2^n signature matrices, $\det(rI - SA) \leq 0$ at $r = 0$.

The following characterization of ρ_0^S shows the striking similarity to the Perron root of nonnegative matrices. For nonnegative A it is (see, for example, Corollary 8.3.3 in [6])

$$\rho(A) = \max_{\substack{x \geq 0 \\ x \neq 0}} \min_{x_i \neq 0} \frac{(Ax)_i}{x_i} \quad \text{for} \quad A \geq 0.$$

Removing the sign restriction for A we have the following characterization of ρ_0^S.

Theorem 3.2. *Suppose A is a real $n \times n$ matrix. Then*

$$\rho_0^S(A) = \max_{\substack{x \in \mathbf{R}^n \\ x \neq 0}} \min_{x_i \neq 0} \left| \frac{(Ax)_i}{x_i} \right|.$$

Proof. Let S be a signature matrix such that $SAz = \lambda z, 0 \neq z \in \mathbf{R}^n$ with $|\lambda| = \rho_0^S(A)$. Then

$$\min_{z_i \neq 0} \left| \frac{(Az)_i}{z_i} \right| = |\lambda| = \rho_0^S(A),$$

and

$$\rho_0^S(A) \le \max_{\substack{x\in\mathbf{R}^n\\ x\ne 0}} \min_{x_i\ne 0} \left|\frac{(Ax)_i}{x_i}\right|.$$

Conversely, Theorem 3.1 implies for any nonzero vector x

$$\rho_0^S(A) \ge \min_{x_i\ne 0} \left|\frac{(Ax)_i}{x_i}\right|.$$

The theorem is proved. □

Theorem 3.1 implies, similar to Perron-Frobenius theory, inheritance of ρ_0^S: going to a principal submatrix cannot increase the sign-real spectral radius. To see this let $\tilde{A}$ be a principal submatrix, and Let $0 \ne z \in \mathbf{R}^k$ with $\rho_0^S(\tilde{A}) = \min_{z_i\ne 0} \left|\frac{(\tilde{A}z)_i}{z_i}\right|$. Augmenting z with zeros to a vector $x \in \mathbf{R}^n$ and using Theorem 3.1 proves the following result.

Corollary 3.3. *Suppose A is a real $n \times n$ matrix. Then ρ_0^S has the inheritance property:*

$$\rho_0^S(A) \ge \rho_0^S(\tilde{A}) \quad \text{for any principal submatrix } \tilde{A} \text{ of } A. \tag{16}$$

Combining (9) and Theorem 3.2 yields

$$\sigma(A,E) = \frac{1}{\max\limits_{|\tilde{E}|\le E} \max\limits_{\substack{x\in\mathbf{R}^n\\ x\ne 0}} \min\limits_{x_i\ne 0} \left|\frac{(A^{-1}\tilde{E}x)_i}{x_i}\right|}.$$

That means any $\tilde{E}$ with $|\tilde{E}| \le E$ and any $x \ne 0$ yield an upper bound for $\sigma(A,E)$. Following, we will identify suitable $\tilde{E}$ and x for proving the upper bound in (4).

4. Lower bounds on ρ_0^S. Using (9), any lower bound on ρ_0^S implies an upper bound on $\sigma(A,E)$, and this is what we need to make progress with the right inequality of (4). Corollary 3.3 already gives such lower bounds, among them

$$\rho_0^S(A) \ge \max |A_{ii}|. \tag{17}$$

By using the inheritance property (16), a lower bound on the sign-real spectral radius of any 2×2 principal submatrix of A is a lower bound for the matrix A as well.

Lemma 4.1. *Suppose A is a real 2×2 matrix with* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Then*

$$\rho_0^S(A) \ge \sqrt{|bc|}. \tag{18}$$

Proof. Using (10) we may assume without loss of generality $bc \ge 0$. Then the eigenvalues $(a+d \pm \sqrt{(a-d)^2+4bc})/2$ of A are both real, and (18) follows. □

As a consequence of Lemma 4.1 and Corollary 3.3, we note

$$\rho_0^S(A) \ge \max \sqrt{|A_{ij} \cdot A_{ji}|} \quad \text{for all } i,j. \tag{19}$$

This concept can be extended to $k \times k$ submatrices as follows. A not necessarily ordered subset $\omega = (\omega_1, ..., \omega_k)$ of $k \ge 1$ mutually distinct integers out of $\{1, ..., n\}$ defines a *cycle*

$$(A_{\omega_1\omega_2}, A_{\omega_2\omega_3}, ..., A_{\omega_k\omega_1})$$

in A. Note that different from other definitions [4], [12] a diagonal element A_{ii} is a cycle of length 1. The geometric mean of a *cycle product* $A_\omega := \prod_{i=1}^{k} A_{\omega_i\omega_{i+1}}$ with $\omega_{k+1} := \omega_1$ is

$$\left|\prod A_\omega\right|^{1/|\omega|} = \left|\prod_{i=1}^{k} A_{\omega_i\omega_{i+1}}\right|^{1/k} \quad \text{with } \omega_{k+1} := \omega_1 \text{ and } |\omega| := k.$$

The lower bounds (17) and (19) are geometric means of cycle products of length 1 and 2, respectively. The aim of this chapter is to derive a similar lower bound depending on the geometric mean of cycles of length greater than 2. Before we proceed on this we need the following technical lemma.

LEMMA 4.2. *Suppose L is a strictly lower triangular matrix, suppose D is a diagonal matrix, and suppose x is a vector. Then there exists a signature matrix S with*

$$|(L+D)Sx| \geq |DSx|. \tag{20}$$

Proof. We show by induction over the rows k that $|(L+D)Sx|_k \geq |DSx|_k$ for a suitable signature matrix S. For $k = 1$ there is nothing to prove because L is strictly lower triangular. Suppose there exists some signature matrix S satisfying (20) for rows i, $1 \leq i < k$. If $(LSx)_k$ has the same sign as $(DSx)_k$, (20) is already satisfied for $i = k$:

$$|(L+D)Sx|_k = |LSx|_k + |DSx|_k \geq |DSx|_k.$$

Otherwise, set $S_{kk} := -1$. Then (20) is still satisfied for $1 \leq i < k$ because L is strictly lower triangular, and by construction it is also satisfied for $i = k$. The induction finishes the proof. □

In preparation for the general case we start with a lower bound on ρ_0^S depending on cycle products in a special case.

LEMMA 4.3. *Suppose A is an $n \times n$ matrix with $|A_{ij}| \leq 1, 1 \leq i,j \leq n$. For the full cycle $\omega = (1,2,...,n)$ suppose $A_{\omega_i\omega_{i+1}} = 1$ for $1 \leq i \leq n$ and $\omega_{n+1} := \omega_1$. Then*

$$\rho_0^S(A) > (3+2\sqrt{2})^{-1}. \tag{21}$$

Proof. Define the permutation matrix P by $P_{\omega_i\omega_{i+1}} = 1$ for $1 \leq i \leq n$. Then all diagonal elements of P^TA are equal to one. Define the splitting

$$P^TA = L + I + U \tag{22}$$

into the strictly lower triangular matrix L, the identity matrix I, and the strictly upper triangular matrix U. In order to be able to apply Theorem 3.1 we will identify a positive vector $y \in \mathbf{R}^n$ with $|ASy| > (3+2\sqrt{2})^{-1} \cdot |y|$ for some signature matrix S.

For the moment, let $x \in \mathbf{R}^n$ be an arbitrary positive vector. By Lemma 4.2 and (22), there exists a signature matrix S with

$$\begin{aligned} |ASx| &= P \cdot |(L+I+U)Sx| \geq P \cdot \{|(L+I)Sx| - |USx|\} \\ &\geq P \cdot \{|Sx| - |USx|\} = Px - P \cdot |USx|. \end{aligned} \tag{23}$$

By assumption, $|U_{ij}| \le 1$, and therefore $(P \cdot |USx|)_i \le \sum_{\nu=i+2}^{n} x_\nu$ for $1 \le i \le n-1$ and $(P \cdot |USx|)_n \le \sum_{\nu=2}^{n} x_\nu$. Together with (23) it follows that for any positive $x \in \mathbf{R}^n$ there exists a signature matrix S with

$$
(24) \qquad \begin{aligned} |ASx|_i &\ge x_{i+1} - \sum_{\nu=i+2}^{n} x_\nu \quad \text{for } 1 \le i \le n-1, \text{ and} \\ |ASx|_n &\ge x_1 - \sum_{\nu=2}^{n} x_\nu . \end{aligned}
$$

Our specific choice $y = x$ is

$$
y_i := q^i \quad \text{for } 1 \le i \le n \quad \text{with } q := 1 - \sqrt{2}/2 \, .
$$

Then $1 - \sum_{\nu=1}^{n} q^\nu > 1 - q/(1-q) = 2 - 2\sqrt{2}$, and by (24)

$$
\begin{aligned} |ASy|_i \ge y_{i+1} - \sum_{\nu=i+2}^{n} y_\nu > q^i \cdot q \cdot (2-\sqrt{2}) = (3+2\sqrt{2})^{-1} \cdot y_i \text{ for } 1 \le i \le n-1, \\ |ASy|_n \ge y_1 - \sum_{\nu=2}^{n} y_i > q \cdot (2-\sqrt{2}) > (3+2\sqrt{2})^{-1} \cdot y_n \, . \end{aligned}
$$

Hence $|ASy| > (3+2\sqrt{2})^{-1} \cdot |y|$, and Theorem 3.1, $\rho_0^S(A) = \rho_0^S(AS)$ and a continuity argument finish the proof. ∎

With these preparatory lemmas we can prove a lower bound for the sign-real spectral radius using cycle products.

THEOREM 4.4. *Suppose A is an $n \times n$ matrix, and suppose $\omega = (\omega_1, ..., \omega_k)$ is a cycle. Then*

$$
\rho_0^S(A) > (3+2\sqrt{2})^{-1} \cdot \left| \prod A_\omega \right|^{1/|\omega|} .
$$

Proof. Without loss of generality we may assume $\zeta := |\prod A_\omega|^{1/|\omega|}$ to be the maximum geometric mean of the cycle products of A.

Set $B := \zeta^{-1} \cdot A[\omega] \in M_k(\mathbf{R})$ with $A[\omega]$ being the principal submatrix of A with rows and columns in ω. Then $|\prod B_{\tilde{\omega}}| = 1$ for the full cycle $\tilde{\omega} = (1, ..., k)$, and this is the maximum absolute value of a cycle product in B (because otherwise ζ would not be maximal in A). Diagonal similarity transformations leave cycle products invariant. Define the $k \times k$ diagonal matrix D by

$$
D_{ii} := \prod_{\nu=i}^{k} B_{\nu,\nu+1} \quad \text{for} \quad 1 \le i \le k \quad \text{and} \quad B_{k,k+1} := B_{k1}.
$$

A computation using $\left| \prod_{\nu=1}^{k} B_{\nu,\nu+1} \right| = 1$ yields

$$
|(D^{-1}BD)_{i,i+1}| = 1 \quad \text{for} \quad 1 \le i \le k \quad \text{and} \quad k+1 \quad \text{interpreted as } 1.
$$

Hence, the absolute value of all elements of the full cycle $\tilde{\omega} = (1, ..., k)$ of $C := D^{-1}BD$ is equal to 1. Furthermore, it is $|C_{ij}| \le 1$, because otherwise there would exist a cycle product of C and henceforth of B larger than 1 in absolute value. Finally, for a suitable signature matrix S it is $(SC)_{i,i+1} = 1$ for $1 \le i \le k, k+1$ interpreted as 1. Hence $SC = SB^{-1}DB$ satisfies the assumptions of Lemma 4.3. Using (10), (11), (16), the definition of B and Lemma 4.3 yield

$$\rho_0^S(A) \ge \rho_0^S(A[\omega]) = \zeta \cdot \rho_0^S(B) = \zeta \cdot \rho_0^S(SD^{-1}BD) > (3+2\sqrt{2})^{-1} \cdot \zeta.$$

The proof is finished. □

We mention that the constant $(3+2\sqrt{2})^{-1}$ in Theorem 4.4 cannot be replaced by a constant larger than 1/2.

5. Main result. We now turn to the proof of (4). The working horses will be (9), Theorem 3.2 and Theorem 4.4. First, we mention that the right inequality in (4) is sharp up to the constant factor $3+2\sqrt{2}$. This is because for

$$A = \begin{pmatrix} 1 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & & 0 & 1 \\ & & & & 1 & s \end{pmatrix} \quad \text{with} \quad s = (-1)^{n+1} \tag{25}$$

it is

$$\sigma(A, |A|) = \frac{n}{\rho(|A^{-1}|\,|A|)} \quad . \tag{26}$$

This can be seen because on the one hand, $|A^{-1}| \cdot |A| = (1)_{nn}$, the matrix with all 1's, and therefore $\rho(|A^{-1}| \cdot |A|) = n$ for all n. On the other hand, the determinant of A is equal to the sum of two full cycles in A, both being equal to 1 in absolute value, and s is choosen such that $|\det A| = 2$. Therefore, any relative change of the components of A less than 100% cannot produce a singular matrix, henceforth $\sigma(A, |A|) = 1$. Note that the matrix in (25) is symmetric and relative perturbations are used. In [**?**] a similar example with weakly diagonally dominant A is given.

For the proof of the right inequality in (4) let a nonsingular $n \times n$ matrix A and a nonnegative $n \times n$ matrix E be given, and set $B := |A^{-1}| \cdot E$. If $\rho(B) = 0$, then (9) implies $\sigma(A, E)^{-1} = \max_{|\tilde{E}| \le E} \rho_0^S(A^{-1}\tilde{E}) \le \rho(|A^{-1}|E) = 0$, and (4) is satisfied. For the following assume $r := \rho(B) > 0$.

Using the irreducible normal form of the nonnegative matrix B, there is an irreducible $k \times k$ principal submatrix C of B with $\rho(C) = \rho(B) = r$. If B itself is irreducible, then $C = B$. Let x be the positive eigenvector of C to the positive Perron root $\rho(C) = r$. Let diagonal D have the elements of x in the diagonal, and denote by (1) the vector with all components equal to 1. Then

$$D^{-1}CD \cdot (1) = r \cdot (1).$$

Therefore, in each row of the nonnegative matrix $D^{-1}CD$ there exists an element greater or equal to r/k, and there is a cycle $(D^{-1}CD)_{\tilde{\omega}}$, $\tilde{\omega}$ out of $\{1, ..., k\}$, with

$$r/k \le \left|\prod (D^{-1}CD)_{\tilde{\omega}}\right|^{1/|\tilde{\omega}|} = \left|\prod C_{\tilde{\omega}}\right|^{1/|\tilde{\omega}|}$$

(recall that diagonal transformations leave cycle products invariant). The cycle $\tilde{\omega}$ in the principal submatrix C of B defines a cycle B_ω in B, $\omega = (\omega_1, ..., \omega_k)$ out of $\{1, ..., n\}$, with

$$(27) \qquad r/n \le r/k \le \left|\prod C_{\tilde{\omega}}\right|^{1/|\tilde{\omega}|} = \left|\prod B_\omega\right|^{1/|\omega|} = \left|\prod (|A^{-1}|E)_\omega\right|^{1/|\omega|}.$$

The elements of the cycle $(|A^{-1}|E)_\omega$ are located in mutually different rows and columns, respectively. Therefore, the signs of the elements of an $n \times n$ matrix F with $|F| = E$ can be chosen such that $A^{-1}F$ and $|A^{-1}|E$ have coinciding elements in the cycle ω:

$$(A^{-1}F)_{\omega_i\omega_{i+1}} = (|A^{-1}|E)_{\omega_i\omega_{i+1}} \quad \text{for} \quad 1 \le i \le k,\ \omega_{k+1} := \omega_1.$$

Hence, (9) together with Theorem 4.4 and (27) yields

$$\begin{aligned}\sigma(A,E)^{-1} &= \max_{|\tilde{E}| \le E} \rho_0^S(A^{-1}\tilde{E}) \ge \rho_0^S(A^{-1}F) > (3+2\sqrt{2})^{-1} \cdot \left|\prod (A^{-1}F)_\omega\right|^{1/|\omega|} \\ &\ge (3+2\sqrt{2})^{-1} \cdot r/n = \frac{\rho(|A^{-1}|E)}{(3+2\sqrt{2})n}.\end{aligned}$$

This proves the right inequality of (4) and together with (25), (26) we have the following proposition.

PROPOSITION 5.1. *Suppose A is a nonsingular $n \times n$ matrix, and suppose E is a nonnegative $n \times n$ matrix. Then*

$$\frac{1}{\rho(|A^{-1}|E)} \le \sigma(A,E) < \frac{(3+2\sqrt{2}) \cdot n}{\rho(|A^{-1}|E)}.$$

The constant $3 + 2\sqrt{2}$ cannot be replaced by 1.

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