

Guaranteed Inclusions for the Complex Generalized Eigenproblem

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Abstract — Zusammenfassung

Guaranteed Inclusions for the Complex Generalized Eigenproblem. A method is described which produces guaranteed bounds for a solution of the generalized complex eigenproblem. The method extends a similar approach for general systems of nonlinear equations to the special case of complex pencils, where under weaker assumptions stronger assertions can be proved.

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Einschließung der Lösung für das allgemeine, komplexe Eigenproblem. Es wird eine Methode zur Berechnung garantierter Schranken für die Lösung des komplexen allgemeinen Eigenproblems beschrieben. Die Methode erweitert einen ähnlichen Ansatz für allgemeine nichtlineare Gleichungssysteme in der Art, daß für den vorliegenden speziellen Fall weitgehende Folgerungen aus schwächeren Voraussetzungen gezogen werden können.

0. Introduction

Let T be one of the sets \mathbb{C} (complex numbers), \mathbb{C}^n (complex vectors with n components) or $\mathbb{C}^{n \times n}$ (complex square matrices with n rows and columns). In the power set $\mathbb{P}T$ operations are defined by

$$A, B \in \mathbb{P}T: A * B := \{a * b \mid a \in A, b \in B\} \quad \text{for } * \in \{+, -, \cdot, / \}$$

with obvious restrictions for $/$. The order relation in \mathbb{R} is extended to a partial ordering in \mathbb{C} and extended componentwise in \mathbb{C}^n and $\mathbb{C}^{n \times n}$.

$\overset{\circ}{A}$ denotes the interior of a set A , I_m the $m \times m$ identity matrix, e'_k the k -th unit (row-)vector.

Sets occurring in an expression several times are treated *independently*, which means for example for $A, B \in \mathbb{P}T$

$$A + B \cdot A = \{a_1 + b \cdot a_2 \mid a_1, a_2 \in A \text{ and } b \in B\}.$$

This is fundamental for the following. In practical implementations this will always be satisfied automatically.

The sets $\mathbb{I}T \subseteq \mathbb{P}T$ of intervals over T are defined by

$$[A, B] \in \mathbb{I}T : \Leftrightarrow \{x \in T \mid A \leq x \leq B\} \quad \text{for } A, B \in T.$$

Power set operations in $\mathbb{I}T$ are induced by those in $\mathbb{P}T$ whereas interval operations \diamond are defined by

$$A \in \mathbb{I}T_1, B \in \mathbb{I}T_2 : A \diamond B := \cap \{C \in \mathbb{I}T_3 \mid A * B \subseteq C\},$$

where T_1, T_2, T_3 are either one of the sets \mathbb{C}, \mathbb{C}^n or $\mathbb{C}^{n \times n}$ such that for $X \in T_1, Y \in T_2$, $X * Y$ is well-defined and $X * Y \in T_3$.

These interval operations are welldefined (see [11], [12], [2]). Intervals the bounds of which are floating-point numbers are defined in a similar way as well as operations between those. For more details see [2], [4] or [15]. [4] gives a very nice introduction to inclusion algorithms. For the following discussion it suffices to know that operations between intervals with floating-point bounds are welldefined, are quickly executable on digital computers and give sharp bounds (in terms of intervals) of the solution set.

One purpose of the following discussions will be to formulate theorems allowing to calculate sharp inclusions of the solution by diminishing overestimations introduced by interval calculations. Thereby, mathematically equivalent formulations may differ vastly in the corresponding practical results.

1. First Results

For $A, B \in \mathbb{C}^{n \times n}$ the problem will be discussed finding inclusions of an eigenvector/eigenvalue pair of the pencil $A - \lambda B$. First we derive a theorem which follows from a general theorem for the inclusion of the solution of systems of nonlinear equations (see [15]).

We use a normalization

$$e'_k \cdot x = \zeta \tag{1.1}$$

for the eigenvector x with some $0 \neq \zeta \in \mathbb{C}$. Other normalizations are possible as well. The problem is rewritten to find a zero of a nonlinear system $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ with

$$f \begin{pmatrix} x \\ \lambda \end{pmatrix} := \begin{pmatrix} Ax - \lambda Bx \\ e'_k x - \zeta \end{pmatrix}, \tag{1.2}$$

where $x \in \mathbb{C}^n, \zeta \in \mathbb{C}$. The Jacobian J of f computes directly as

$$J := \begin{pmatrix} A - \lambda B & -Bx \\ e'_k & 0 \end{pmatrix}. \tag{1.3}$$

A nonlinear system similar to (1.2) has been discussed by Krawczyk [7]. We use these ideas and the principles of inclusion algorithms [15] extended to the generalized eigenproblem.

The following ideas and corresponding algorithms can be regarded as the extension of a traditional numerical algorithm to an interval-type algorithm providing results which are based on a (good) floating-point approximation and which are guaranteed to be correct.

With these preliminary remarks we can prove the following theorem.

Theorem 1: *Let $A, B \in \mathbb{C}^{n \times n}$, $R \in \mathbb{C}^{(n+1) \times (n+1)}$, $\tilde{x} \in \mathbb{C}^n$, $\tilde{\lambda}, \zeta \in \mathbb{C}$ with $\zeta \neq 0$. Let $X \in \mathbb{P}\mathbb{C}^n$, $\Lambda \in \mathbb{P}\mathbb{C}$ be nonempty, compact and convex sets with $\tilde{x} \in X$ and $\tilde{\lambda} \in \Lambda$ and define*

$$Z := \begin{pmatrix} \tilde{x} \\ \tilde{\lambda} \end{pmatrix} - R \cdot \begin{pmatrix} A\tilde{x} - \tilde{\lambda}B\tilde{x} \\ e'_k \tilde{x} - \zeta \end{pmatrix} \tag{1.4}$$

and

$$\Delta := \left\{ I_{n+1} - R \cdot \begin{pmatrix} A - \Lambda B & -BX \\ e'_k & 0 \end{pmatrix} \right\} \cdot \begin{pmatrix} X - \tilde{x} \\ \Lambda - \tilde{\lambda} \end{pmatrix}. \tag{1.5}$$

If

$$Z + \Delta \subseteq \text{interior} \begin{pmatrix} X \\ \Lambda \end{pmatrix} \tag{1.6}$$

then there exist some $\hat{x} \in X$, $\hat{\lambda} \in \Lambda$ with $A\hat{x} = \hat{\lambda}B\hat{x}$ and $e'_k \hat{x} = \zeta$.

Proof: In every ε -neighborhood of R there exists a nonsingular matrix. Therefore, by (1.6) some nonsingular \tilde{R} exists satisfying

$$\tilde{Z} + \tilde{\Delta} \subseteq \text{interior} \begin{pmatrix} X \\ \Lambda \end{pmatrix} \tag{1.7}$$

where \tilde{Z} and $\tilde{\Delta}$ are defined similar to Z and Δ by replacing R by \tilde{R} . Regarding $\tilde{\lambda} \in \Lambda$ and using the definition (1.2) of f and (1.7) yields for every $x \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ with $x \in X$, $\lambda \in \Lambda$ after short computation

$$\begin{aligned} \begin{pmatrix} x \\ \lambda \end{pmatrix} - \tilde{R} \cdot f \begin{pmatrix} x \\ \lambda \end{pmatrix} &= \begin{pmatrix} x \\ \lambda \end{pmatrix} - \tilde{R} \cdot \begin{pmatrix} Ax - \lambda Bx \\ e'_k x - \zeta \end{pmatrix} = \\ \begin{pmatrix} \tilde{x} \\ \tilde{\lambda} \end{pmatrix} - \tilde{R} \cdot \begin{pmatrix} A\tilde{x} - \tilde{\lambda}B\tilde{x} \\ e'_k \tilde{x} - \zeta \end{pmatrix} + \left\{ I_{n+1} - \tilde{R} \cdot \begin{pmatrix} A - \tilde{\lambda}B & -Bx \\ e'_k & 0 \end{pmatrix} \right\} \cdot \begin{pmatrix} x - \tilde{x} \\ \lambda - \tilde{\lambda} \end{pmatrix} &\subseteq \\ \begin{pmatrix} \tilde{x} \\ \tilde{\lambda} \end{pmatrix} - \tilde{R} \cdot \begin{pmatrix} A\tilde{x} - \tilde{\lambda}B\tilde{x} \\ e'_k \tilde{x} - \zeta \end{pmatrix} + \left\{ I_{n+1} - \tilde{R} \cdot \begin{pmatrix} A - \Lambda B & -BX \\ e'_k & 0 \end{pmatrix} \right\} \cdot \begin{pmatrix} X - \tilde{x} \\ \Lambda - \tilde{\lambda} \end{pmatrix} &\subseteq \\ \text{interior} \begin{pmatrix} X \\ \Lambda \end{pmatrix}. & \end{aligned} \tag{1.8}$$

Brouwer's Fixed Point Theorem yields the existence of some $\hat{x} \in X$, $\hat{\lambda} \in \Lambda$ with

$$\begin{pmatrix} \hat{x} \\ \hat{\lambda} \end{pmatrix} - \tilde{R} \cdot f \begin{pmatrix} \hat{x} \\ \hat{\lambda} \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{\lambda} \end{pmatrix}$$

and the nonsingularity of \tilde{R} and the definition of f finishes the proof. □

The theorem for nonlinear equations in [15] yields, moreover, the uniqueness of the pair $\hat{x}, \hat{\lambda}$. It should be mentioned that Theorem 1 can be proved without assuming X and Λ to be convex using finer arguments. The aim of the next chapter is to prove the uniqueness of the eigenvalue/eigenvector pair as well as the individual uniqueness of the eigenvector and eigenvalue within X and Λ , respectively.

2. Main Results

For the succeeding discussions we use the following abbreviations.

Let $A, B \in \mathbb{C}^{n \times n}$; $R \in \mathbb{C}^{(n+1) \times (n+1)}$; $\tilde{x} \in \mathbb{C}^n$;
 $\tilde{\lambda}, \zeta \in \mathbb{C}$, $G: \mathbb{P}\mathbb{C}^{n+1} \rightarrow \mathbb{P}\mathbb{C}^{n+1}$ defined by

$$G \begin{pmatrix} Y \\ M \end{pmatrix} := Z + \{I_{n+1} - R \cdot S(Y)\} \cdot \begin{pmatrix} Y - \tilde{x} \\ M - \tilde{\lambda} \end{pmatrix}$$

with $Y \in \mathbb{P}\mathbb{C}^n$, $M \in \mathbb{P}\mathbb{C}$ and

$$Z := \begin{pmatrix} \tilde{x} \\ \tilde{\lambda} \end{pmatrix} - R \cdot \begin{pmatrix} A\tilde{x} - \tilde{\lambda}B\tilde{x} \\ e'_k \tilde{x} - \zeta \end{pmatrix} \in \mathbb{C}^{n+1} \quad (2.1)$$

and

$$S(Y) := \begin{pmatrix} A - \tilde{\lambda}B & -BY \\ e'_k & 0 \end{pmatrix} \in \mathbb{P}\mathbb{C}^{(n+1) \times (n+1)}$$

k is a fixed integer between 1 and n , all operations in use are the power set operations.

The problem is to find inclusions of an eigenvalue/eigenvector pair of the pencil $A - \lambda B$. In (2.1) there are no assumptions on any of the used entities $A, B, R, \tilde{x}, \tilde{\lambda}$ and ζ .

We will use the fact that for $x \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ and f defined by (1.2),

$$G \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} x \\ \lambda \end{pmatrix} - R \cdot f \begin{pmatrix} x \\ \lambda \end{pmatrix}, \quad (2.2)$$

as short computation yields. We first state the main result and give the proof in several steps.

Theorem 2: *With the abbreviations (2.1) let $\zeta \neq 0$, $X \in \mathbb{P}\mathbb{C}^n$ and $\Lambda \in \mathbb{P}\mathbb{C}$ both be nonempty, compact and connected and suppose*

$$G \begin{pmatrix} X \\ \Lambda \end{pmatrix} \subseteq \text{interior} \begin{pmatrix} X \\ \Lambda \end{pmatrix}. \quad (2.3)$$

Then

- there exists one and only one eigenvector \hat{x} of the pencil $Ax = \lambda Bx$ normalized to $e'_k \cdot \hat{x} = \zeta$ satisfying $\hat{x} \in X$,
- there exists one and only one eigenvalue $\hat{\lambda}$ of the pencil $Ax = \lambda Bx$ satisfying $\hat{\lambda} \in \Lambda$,
- \hat{x} and $\hat{\lambda}$ satisfy $A\hat{x} = \hat{\lambda}B\hat{x}$.

First we proof the existence and uniqueness of an eigenvector/eigenvalue pair within (X, Λ) . Note that the assumptions in Theorem 2 are weaker than in Theorem 1 because according to (2.1), $S(Y)$ contains only $A - \tilde{\lambda}B$ instead of $A - \Lambda \cdot B$ in the upper left corner and $\tilde{x}, \tilde{\lambda}$ are not supposed to be elements of X, Λ , respectively. Furthermore, X and Λ are not supposed to be convex but only connected. For this purpose we need the following important technical lemma.

Lemma 3: *With the assumptions of Theorem 2 the matrix R and every matrix $Q \in S(X)$ are nonsingular.*

Proof: Follows by Theorem 5 in [16] and (2.1). □

Lemma 4: *With the assumptions of Theorem 2 there exists one and only one eigenvector/eigenvalue pair of $Ax - \lambda Bx$ subject to the normalization $e'_k x = \zeta$ within (X, Λ) , i.e.*

$$\exists \hat{x} \in X \exists \hat{\lambda} \in \Lambda : \hat{x} \neq 0 \text{ and } A\hat{x} = \hat{\lambda}B\hat{x} \text{ and } y \in X, \mu \in \Lambda, e'_k y = \zeta \text{ and } Ay = \mu By \text{ implies } y = \hat{x}, \mu = \hat{\lambda}.$$

Proof: Using (2.2) and (2.3) yields

$$\left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} - R \cdot f \begin{pmatrix} x \\ \lambda \end{pmatrix} \mid x \in X, \lambda \in \Lambda \right\} \subseteq G \begin{pmatrix} X \\ \Lambda \end{pmatrix} \subseteq \text{interior} \begin{pmatrix} X \\ \Lambda \end{pmatrix}$$

implying the existence of a fixed point $\hat{x} \in X, \hat{\lambda} \in \Lambda$ with

$$R \cdot f \begin{pmatrix} \hat{x} \\ \hat{\lambda} \end{pmatrix} = 0. \tag{2.4}$$

Using Lemma 3 yields $A\hat{x} = \hat{\lambda}B\hat{x}$ and $e'_k \hat{x} = \zeta$ and therefore $\hat{x} \neq 0$. For the proof of the uniqueness of the eigenvector/eigenvalue pair $(\hat{x}, \hat{\lambda})$ we assume the existence of $x, y \in X$ and $\lambda, \mu \in \Lambda$ with $e'_k x = e'_k y = \zeta$ and

$$Ax = \lambda Bx \quad \text{and} \quad Ay = \mu By \quad \text{and} \quad x \neq y. \tag{2.5}$$

In the following we need the nonsingularity of every matrix $Q(z), z \in X$ defined by

$$Q(z) := \begin{pmatrix} A - \tilde{\lambda}B & -Bz \\ e'_k & 0 \end{pmatrix} \tag{2.6}$$

which follows by Lemma 3. Assume, $\tilde{\lambda}$ is an eigenvalue, i.e. there exists some $v \in \mathbb{C}^n$ with $Av = \tilde{\lambda}Bv, v \neq 0$. If $e'_k \cdot v = 0$ then using (2.6) we have

$$Q(x) \cdot \begin{pmatrix} v \\ 0 \end{pmatrix} = 0,$$

which is a contradiction because $v \neq 0$ and $Q(x)$ is not singular. If $e'_k \cdot v \neq 0$, then we may assume w.l.o.g. $e'_k \cdot v = \zeta$ and therefore

$$Q(x) \cdot \begin{pmatrix} v-x \\ \tilde{\lambda}-\lambda \end{pmatrix} = 0 \quad \text{and} \quad Q(y) \cdot \begin{pmatrix} v-y \\ \tilde{\lambda}-\mu \end{pmatrix} = 0.$$

Now $v \neq x$ or $v \neq y$ because of $x \neq y$ showing the existence of a nonzero vector in the kernel of $Q(x)$ or $Q(y)$, which is a contradiction. Therefore, $\tilde{\lambda}$ is no eigenvalue and especially,

$$\tilde{\lambda} \neq \lambda \quad \text{and} \quad \tilde{\lambda} \neq \mu. \quad (2.7)$$

If $\lambda = \mu$ then short computation yields

$$f \left(\begin{array}{c} x + \delta(y-x) \\ \lambda \end{array} \right) = 0 \quad \text{for every } \delta \in \mathbb{C}.$$

Then (2.2) shows that $(x + \delta(y-x), \lambda)'$ would be a fixed point of G for every $\delta \in \mathbb{C}$ contradicting (2.3), because by $y \neq x$ we have $x + \delta^*(y-x) \in \partial X$ for some $\delta^* \in \mathbb{R}$. Therefore, we may assume $\lambda \neq \mu$ for the following.

Let $\delta \in \mathbb{C}$ and define

$$N(\delta) := (1-\delta) \cdot (\mu - \tilde{\lambda}) + \delta(\lambda - \tilde{\lambda}). \quad (2.8)$$

Then

$$N(\delta) = 0 \Leftrightarrow \delta(\lambda - \mu) = \tilde{\lambda} - \mu. \quad (2.9)$$

We assume for the moment $(\lambda - \mu) \delta \neq \tilde{\lambda} - \mu$ and define

$$w(\delta) := x + \frac{\delta(\lambda - \tilde{\lambda})}{N(\delta)} \cdot (y-x) \in \mathbb{C}^n$$

and

$$\sigma(\delta) := \tilde{\lambda} + \frac{\mu - \tilde{\lambda}}{N(\delta)} \cdot (\lambda - \tilde{\lambda}) \in \mathbb{C}. \quad (2.10)$$

Moreover, let $f_z : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ be defined by

$$f_z \left(\begin{array}{c} w \\ \sigma \end{array} \right) := \left(\begin{array}{c} w \\ \sigma \end{array} \right) - R \cdot \left(\begin{array}{c} (A - \tilde{\lambda}B)w - (\sigma - \tilde{\lambda})Bz \\ e'_k w - \zeta \end{array} \right) \quad \text{for } z \in \mathbb{C}^n \quad (2.11)$$

and $w \in \mathbb{C}^n$, $\sigma \in \mathbb{C}$.

Next we show that $(w(\delta), \sigma(\delta))'$ is a fixed point of $f_{x+\delta(y-x)}$.

$$\begin{aligned} & (A - \tilde{\lambda}B) \cdot w(\delta) - (\sigma(\delta) - \tilde{\lambda})B \cdot w(\delta) = \\ & (A - \tilde{\lambda}B) \left\{ x + \frac{\delta(\lambda - \tilde{\lambda})}{N(\delta)} \cdot (y-x) \right\} - \frac{\mu - \tilde{\lambda}}{N(\delta)} (\lambda - \tilde{\lambda}) \cdot B \cdot (x + \delta(y-x)) = \\ & (\lambda - \tilde{\lambda})Bx + \frac{\delta(\lambda - \tilde{\lambda})}{N(\delta)} ((\mu - \tilde{\lambda})By - (\lambda - \tilde{\lambda})Bx) - \frac{\mu - \tilde{\lambda}}{N(\delta)} (\lambda - \tilde{\lambda}) \cdot B \cdot ((1-\delta)x + \delta y) = \\ & \left\{ \lambda - \tilde{\lambda} - \frac{\delta(\lambda - \tilde{\lambda})^2}{N(\delta)} - \frac{\mu - \tilde{\lambda}}{N(\delta)} (\lambda - \tilde{\lambda})(1-\delta) \right\} \cdot Bx + \\ & \left\{ \frac{\delta(\lambda - \tilde{\lambda})}{N(\delta)} (\mu - \tilde{\lambda}) - \frac{\mu - \tilde{\lambda}}{N(\delta)} (\lambda - \tilde{\lambda}) \delta \right\} By = \\ & (\lambda - \tilde{\lambda}) \cdot \{ 1 - N(\delta)^{-1} \cdot (\delta(\lambda - \tilde{\lambda}) + (\mu - \tilde{\lambda})(1-\delta)) \} \cdot Bx = 0. \end{aligned} \quad (2.12)$$

Moreover, $e'_k \cdot w(\delta) = \zeta$ implies with (2.11)

$$f_{x+\delta(y-x)} \begin{pmatrix} w(\delta) \\ \sigma(\delta) \end{pmatrix} = \begin{pmatrix} w(\delta) \\ \sigma(\delta) \end{pmatrix} \text{ for all } \delta \in \mathbb{C} \text{ with } (\lambda - \mu)\delta \neq \tilde{\lambda} - \mu. \quad (2.13)$$

(x, λ) and (y, μ) are fixed points of G by (2.2). With (2.3) and $x, y \in X$ this shows $x, y \in X \setminus \partial X$ and the existence of some $\delta_1, \delta_2 \in \mathbb{R}$ with $\delta_1 < 0 < 1 < \delta_2$ and the property that $\delta \in \mathbb{R}$ and $\delta_1 \leq \delta \leq \delta_2$ implies $x + \delta(y - x) \in X$ where $x + \delta_1(y - x) \in \partial X$ and $x + \delta_2(y - x) \in \partial X$.

By (2.3) we have

$$\left\{ f_{x+\delta(y-x)} \begin{pmatrix} u \\ \eta \end{pmatrix} \mid u \in X, \eta \in \Lambda \right\} \subseteq \text{interior} \begin{pmatrix} X \\ \Lambda \end{pmatrix} \text{ for all } \delta_1 \leq \delta \leq \delta_2. \quad (2.14)$$

$w(0) = x$ and $w(1) = y$ because $\lambda \neq \tilde{\lambda}$ and therefore, together with (2.14),

$$w(\delta) \in X \setminus \partial X \text{ for all } \delta_1 \leq \delta \leq \delta_2 \text{ and } (\lambda - \mu)\delta \neq \tilde{\lambda} - \mu \quad (2.15)$$

because $w(\delta)$ forms for $\delta_1 \leq \delta \leq \delta_2$ a connected curve which, by (2.3) and (2.13), cannot intersect the boundary of X . Suppose $(\tilde{\lambda} - \mu)/(\lambda - \mu) \in [\delta_1, \delta_2]$. Then $x \neq y$, (2.9) and (2.10) show that $w(\delta)$ tends to infinity for $\delta \rightarrow (\tilde{\lambda} - \mu)/(\lambda - \mu)$. This implies the existence of some $w(\delta^*) \in \partial X$ for $\delta_1 \leq \delta^* \leq \delta_2$ and $\delta^* \neq (\tilde{\lambda} - \mu)/(\lambda - \mu)$ contradicting (2.15).

Therefore,

$$\frac{\tilde{\lambda} - \mu}{\lambda - \mu} \notin [\delta_1, \delta_2] \text{ and } w(\delta) \text{ is well-defined for } \delta_1 \leq \delta \leq \delta_2. \quad (2.16)$$

Together with (2.15) and (2.8) this implies

$$\delta_1 < \frac{\delta(\lambda - \tilde{\lambda})}{(1 - \delta)(\mu - \tilde{\lambda}) + \delta(\lambda - \tilde{\lambda})} < \delta_2 \text{ for all } \delta_1 \leq \delta \leq \delta_2. \quad (2.17)$$

$N(\delta)$ defined by (2.8) has constant sign for all $\delta_1 \leq \delta \leq \delta_2$. Suppose $N(\delta) > 0$. Then the left inequality in (2.17) yields for $\delta = \delta_1$ using $\delta_1 < 0$

$$\begin{aligned} \delta_1 \cdot \{(1 - \delta_1)(\mu - \tilde{\lambda}) + \delta_1(\lambda - \tilde{\lambda})\} &< \delta_1(\lambda - \tilde{\lambda}) \Rightarrow \\ \mu - \tilde{\lambda} + \delta_1(\lambda - \mu) &> \lambda - \tilde{\lambda} \Rightarrow \delta_1(\lambda - \mu) > \lambda - \mu \Rightarrow \lambda < \mu. \end{aligned}$$

The right inequality in (2.17) yields for $\delta = \delta_2$ using $\delta_2 > 1$

$$\lambda - \tilde{\lambda} < \mu - \tilde{\lambda} + \delta_2(\lambda - \mu) \Rightarrow \lambda - \mu < \delta_2(\lambda - \mu) \Rightarrow \lambda > \mu.$$

For $N(\delta) < 0$ the left inequality in (2.17) for $\delta = \delta_1$ and the right inequality in (2.17) for $\delta = \delta_2$ yield the same contradiction which therefore demonstrates the incorrectness of assumption (2.5) and proves Lemma 4. □

Next we prove the individual uniqueness of the eigenvalue in Λ .

Lemma 5: *With the assumptions of Theorem 2 let μ be an eigenvalue of $Ax - \lambda Bx$ with $\mu \in \Lambda$. Then every eigenvector y corresponding to μ can be normalized to $e'_k y = \zeta$ and lies in X .*

Proof: Let $y \in \mathbb{C}^n$ be an eigenvector corresponding to μ . Define $g : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ by

$$g \begin{pmatrix} w \\ \sigma \end{pmatrix} := Z + \left\{ I_{n+1} - R \cdot \begin{pmatrix} A - \sigma B & -B\tilde{x} \\ e'_k & 0 \end{pmatrix} \right\} \cdot \begin{pmatrix} w - \tilde{x} \\ \sigma - \tilde{\lambda} \end{pmatrix},$$

for $w \in \mathbb{C}^n$, $\sigma \in \mathbb{C}$. Then short computation yields

$$g \begin{pmatrix} w \\ \sigma \end{pmatrix} = G \begin{pmatrix} w \\ \sigma \end{pmatrix} \quad \text{for all } w \in \mathbb{C}^n, \sigma \in \mathbb{C}$$

and therefore by (2.3)

$$\left\{ g \begin{pmatrix} x \\ \lambda \end{pmatrix} \mid x \in X, \lambda \in \Lambda \right\} \subseteq \text{interior} \begin{pmatrix} X \\ \Lambda \end{pmatrix}.$$

Applying Theorem 5 in [16] yields the nonsingularity of every matrix

$$P(\sigma) := \begin{pmatrix} A - \sigma B & -B\tilde{x} \\ e'_k & 0 \end{pmatrix} \quad \text{for all } \sigma \in \mathbb{C} \text{ with } \sigma \in \Lambda.$$

Then $P(\mu) \cdot (y, 0)$ must be nonzero and therefore $e'_k \cdot y \neq 0$. W.l.o.g. we assume $e'_k y = \zeta$.

Define $g_\mu : \mathbb{C}^n \rightarrow \mathbb{C}^n$ by

$$\begin{pmatrix} g_\mu(t) \\ v \end{pmatrix} = G \begin{pmatrix} t \\ \mu \end{pmatrix} \quad \text{for } t \in \mathbb{C}^n, v \in \mathbb{C}. \quad (2.18)$$

Then by (2.3), g_μ maps X into itself, is continuous and affine. By Theorem 11 in [16] there exists some

$$z \in X \text{ with } g_\mu(z) = z. \quad (2.19)$$

Let

$$G \begin{pmatrix} z \\ \mu \end{pmatrix} = \begin{pmatrix} z \\ \mu^* \end{pmatrix} \quad \text{for some } \mu^* \in \mathbb{C}. \quad (2.20)$$

Suppose $\mu \neq \mu^*$. Then we define $h : \mathbb{C} \rightarrow \mathbb{C}^n$ by

$$h(v) := \xi \cdot z + (1 - \xi) \cdot y \quad \text{with} \quad \xi := \frac{\mu - v}{\mu - \mu^*} \quad (2.21)$$

and z from (2.19). By (2.20), (2.2) and (1.2),

$$G \begin{pmatrix} z \\ \mu \end{pmatrix} = \begin{pmatrix} z \\ \mu^* \end{pmatrix} = \begin{pmatrix} z \\ \mu \end{pmatrix} - R \cdot \begin{pmatrix} Az - \mu Bz \\ e'_k z - \zeta \end{pmatrix}. \quad (2.22)$$

Furthermore, by (2.21) and (2.22)

$$\begin{aligned} R \cdot \begin{pmatrix} A \cdot h(v) - \mu B h(v) \\ e'_k h(v) - \zeta \end{pmatrix} &= R \cdot \begin{pmatrix} (A - \mu B) \xi z \\ e'_k \xi z + e'_k (1 - \xi) y - \zeta \end{pmatrix} = \\ & \xi \cdot R \cdot \begin{pmatrix} (A - \mu B) z \\ e'_k z - \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ \mu - \mu^* \end{pmatrix}. \end{aligned} \quad (2.23)$$

This implies

$$G \begin{pmatrix} h(v) \\ \mu \end{pmatrix} = \begin{pmatrix} h(v) \\ \mu \end{pmatrix} - \begin{pmatrix} 0 \\ \mu - \mu^* \end{pmatrix} = \begin{pmatrix} h(v) \\ \mu^* \end{pmatrix} \quad \text{for all } v \in \mathbb{C}. \quad (2.24)$$

h is continuous in v and for $\varepsilon \in \mathbb{R}$

$$h(v + \varepsilon) - h(v) = \frac{-\varepsilon}{\mu - \mu^*} z + \frac{\varepsilon}{\mu - \mu^*} \cdot y.$$

We still suppose $\mu \neq \mu^*$ implying $z \neq y$ by (2.20) and therefore $|h(v)| \rightarrow \infty$ for $v \rightarrow \infty$. By (2.24) $h(v)$ is a fixed point of g_μ for every $v \in \mathbb{C}$. This implies the existence of some $v^* \in \mathbb{C}$ with $h(v^*) \in \partial X$ contradicting (2.3). This contradiction shows $\mu = \mu^*$ and with (2.20)

$$G \begin{pmatrix} z \\ \mu \end{pmatrix} = \begin{pmatrix} z \\ \mu \end{pmatrix}.$$

Then short computation yields

$$G \begin{pmatrix} z + \delta(y - z) \\ \mu \end{pmatrix} = \begin{pmatrix} z + \delta(y - z) \\ \mu \end{pmatrix} \quad \text{for every } \delta \in \mathbb{C}$$

implying $y = z$ because otherwise some $\delta^* \in \mathbb{R}$ with $z + \delta^*(y - z) \in \partial X$ would contradict (2.3). From (2.19) we know $y = z \in X$ which finishes the proof. \square

The existence of two eigenvalues within Λ implies the existence of two pairs of eigenvectors/eigenvalues contradicting Lemma 4. Next we prove the individual uniqueness (subject to normalization) of the eigenvector within X .

Lemma 6: *With the assumptions of Theorem 2 let y be an eigenvector of $Ax - \lambda Bx$ with $y \in X$. Then the corresponding eigenvalue $\mu \in \mathbb{C}$ satisfies $\mu \in \Lambda$.*

Proof: Define $g_y : \mathbb{C} \rightarrow \mathbb{C}$ by

$$\begin{pmatrix} z \\ g_y(v) \end{pmatrix} = G \begin{pmatrix} y \\ v \end{pmatrix} \quad \text{for } z \in \mathbb{C}^n, v \in \mathbb{C}. \quad (2.25)$$

Then by (2.3), g_y maps Λ into itself, is continuous and affine. By Theorem 11 in [16] there exists some

$$\sigma \in \mathbb{C} \quad \text{with} \quad g_y(\sigma) = \sigma \quad \text{and} \quad \sigma \in \Lambda. \quad (2.26)$$

Let

$$G \begin{pmatrix} y \\ \sigma \end{pmatrix} = \begin{pmatrix} y^* \\ \sigma \end{pmatrix} \quad \text{for some } y^* \in \mathbb{C}^n. \quad (2.27)$$

Suppose $\mu \neq \sigma$. Then define

$$v(v) := y^* + \eta(y - y^*) \quad \text{with} \quad \eta := \frac{v - \sigma}{\mu - \sigma} \quad (2.28)$$

using σ from (2.26). By (2.27), (2.2) and (1.2)

$$G \begin{pmatrix} y \\ \sigma \end{pmatrix} = \begin{pmatrix} y^* \\ \sigma \end{pmatrix} = \begin{pmatrix} y \\ \sigma \end{pmatrix} - R \cdot \begin{pmatrix} (A - \sigma B)y \\ e'_k y - \zeta \end{pmatrix} = \begin{pmatrix} y \\ \sigma \end{pmatrix} - R \cdot \begin{pmatrix} (\mu - \sigma)By \\ e'_k y - \zeta \end{pmatrix}. \quad (2.29)$$

Furthermore, by (2.28) and for every $v \in C$ we have

$$\begin{aligned} G \begin{pmatrix} y \\ v \end{pmatrix} &= \begin{pmatrix} y \\ v \end{pmatrix} - R \cdot \begin{pmatrix} (A - vB)y \\ e'_k y - \zeta \end{pmatrix} = \begin{pmatrix} y \\ v \end{pmatrix} - \frac{\mu - v}{\mu - \sigma} \cdot R \cdot \begin{pmatrix} (\mu - \sigma)By \\ e'_k y - \zeta \end{pmatrix} \\ &= \begin{pmatrix} y \\ v \end{pmatrix} + \frac{\mu - v}{\mu - \sigma} \cdot \begin{pmatrix} y^* - y \\ 0 \end{pmatrix} = \begin{pmatrix} v(v) \\ v \end{pmatrix}. \end{aligned} \quad (2.30)$$

Therefore, every $v \in \mathbb{C}$ is fixed point of g_y contradicting $g_y(A) \subseteq \text{interior}(A)$ which follows by (2.3). This implies $\mu = \sigma$ and (2.26) finishes the proof. \square

The existence of two eigenvectors within X implies the existence of two pairs of eigenvectors/eigenvalues contradicting Lemma 5.

This finally finishes the proof of Theorem 2.

3. Practical Applications

For the practical application on computers sets may be represented by intervals. All theorems mentioned in the previous chapters can be implemented and be used on digital computers by using intervals (over complex numbers, vectors, matrices) as sets and by substituting each power set operation by its corresponding interval operation for $* \in \{+, -, \cdot, /\}$. A necessary condition for the implementation of an interval arithmetic is a precisely defined floating-point arithmetic or operations with directed roundings [5].

Data afflicted with tolerances may be treated as well. In this case the input data are sets of matrices, in practical computations, for example, interval matrices, and all assertions of Theorem 2 are true for each individual matrix within the tolerances.

In the following we give some numerical examples. The computer in use is an IBM 3090 using the programming package ACRITH [1] for interval operations. The programming environment ABACUS is used (see below). Matrices A and B are chosen to be random Hilbert and Pascal matrices (defined below). The approximations \tilde{x} and $\tilde{\lambda}$ are computed as an eigenvector/eigenvalue pair of $B^{-1}A$; all floating-point computations (including B^{-1}) are performed using LINPACK and EISPACK routines.

Theorem 2 is used as an a posteriori check on the accuracy of \tilde{x} and $\tilde{\lambda}$ by defining X , A , the starting intervals to check on an inclusion, to be $\tilde{x} \cdot (1 \pm \varepsilon)$ and $\tilde{\lambda} \cdot (1 \pm \varepsilon)$, where ε is 10^{-14} . The precision in use is 14 hex or approximately 16 decimal digits on an IBM 3090.

The inclusion algorithm is implemented using ABACUS. This is an interactive programming environment allowing to program in mathematical notation as the following original ABACUS subroutine for an inclusion of the solution of the generalized eigenproblem demonstrates.

```

module(A, B, x, l, X, L)
  n = size(x); <zeta, k> = max(abs(x));
  C = <A - l * B, -B * x; nulls(1, n + 1)>; C(n + 1, k) = 1;
  R = 1/C; Z = ival <x; l> - R * <A * x - l * B * x; x(k) - zeta>;
  C = ival <A - l * B, -B * x; nulls(1, n + 1)>; C(n + 1, k) = 1;
  C = Id - R * C; X = Z; e = 1 + / - 1 e - 14; kk = 0;
  loop
    {Y = X * e; X = Z + C * (Y - <x; l>); kk = kk + 1;
     if X in0 Y or kk = 15 then exit;}
  if X in0 Y then {display 'inclusion'; L = X(n + 1); X = X(1 : n);}
                 else {display 'no inclusion'; X = <>; L = <>;}

```

Fig. 1. ABACUS subroutine for the generalized eigenproblem

Id denotes the identity matrix (automatically adjusting its size), $1 + / - 1 e - 14$ is the interval with left bound $1 - 1 e - 14$ and right bound $1 + 1 e - 14$. The keyword *ival* prior to an expression forces the expression to be evaluated using interval operations. If in an expression at least one interval variable occurs (regardless where) the whole expression is evaluated using interval operations. *in0* denotes inclusion of the left hand side in the interior of the right hand side.

The algorithm works similarly to other inclusion algorithms introduced in [15]. Especially, ε -inflation is used.

In the following tables we display the

1. “number of interval iterations” which is kk in the algorithm above,
2. “minimum number of digits guaranteed”, which means the minimum number of digits coinciding in each left and right bound of the inclusion of eigenvector and eigenvalue, and the
3. “accuracy of the approximation”, which is the number and the minimum number of correct digits of the approximation $\tilde{\lambda}$ and \tilde{x} , respectively.

The random matrices R, S have uniformly distributed components between 0 and 1 and one eigenvector/eigenvalue pair has been chosen randomly. Hilbert matrices and Pascal matrices of dimension n are defined by

$$(H_n)_{ij} := lcm(1, 2, \dots, 2n - 1) / (i + j - 1),$$

$$(P_n)_{ij} := \binom{i + j - 1}{i}.$$

For Hilbert and Pascal matrices all eigenvector/eigenvalue pairs were treated. The following results were achieved.

The computation for $H_8 - \lambda \cdot P_8$ was real because the approximations \tilde{x} and $\tilde{\lambda}$ were real. This may be a reason for the better results than for $P_8 - \lambda \cdot H_8$. The number of digits guaranteed is larger than the number of correct digits of the approximations because of the Newton step within the interval iteration. Such a Newton step performed in floating-point arithmetic would, in general, improve an approximation. However, it should be stressed that a floating-point iteration may very well pretend convergence.

Table 1. Computational results

problem	eigenpair	interval iterations	minimum number of digits guaranteed	accuracy of the approximation	
				\tilde{x}	$\tilde{\lambda}$
$R_{10} - \lambda \cdot S_{10}$		2	16	12	14
$R_{20} - \lambda \cdot S_{20}$		2	16	12	13
$H_8 - \lambda \cdot P_8$	1	1	11	9	9
	2	1	11	9	10
	3	1	10	10	10
	4	1	11	11	10
	5	1	12	10	9
	6	1	14	8	7
	7	1	14	7	6
	8	1	14	6	5
$P_8 - \lambda \cdot H_8$	1	2	8	8	7
	2	2	8	8	7
	3	2	9	9	8
	4	2	11	7	7
	5	2	12	6	6
	6	2	14	6	5
	7	2	14	5	4
	8	2	14	3	3

Finally it should be mentioned that even in “degenerated” cases (singular matrix B), when there are fewer solutions than the dimension of the matrix, the method works. The following example also illustrates the interval iteration.

Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Then $\det(A - \lambda B) = 2\lambda - 2$ and the only solution of the generalized eigenproblem $A - \lambda B$ is $x = (0, 1)$ and $\lambda = 1$. We take $\tilde{x} = x$ and $\tilde{\lambda} = \lambda$. Then

$$R = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & -4 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ -1/2 & 0 & 0 \end{pmatrix}.$$

We take $k=2$, $X := \tilde{x} \pm \varepsilon$ and $\Lambda := \tilde{\lambda} \pm \varepsilon$ and according to (2.1) we obtain

$$\begin{aligned} G \begin{pmatrix} X \\ \Lambda \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \left\{ I_3 - R \cdot \begin{pmatrix} 0 & 0 & -2 \pm 2\varepsilon \\ 1 & 0 & -4 \pm 4\varepsilon \\ 0 & 1 & 0 \end{pmatrix} \right\} \cdot \begin{pmatrix} \pm \varepsilon \\ \pm \varepsilon \\ \pm \varepsilon \end{pmatrix} = \\ &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \pm 8\varepsilon \\ 0 & 0 & 0 \\ 0 & 0 & \pm \varepsilon \end{pmatrix} \begin{pmatrix} \pm \varepsilon \\ \pm \varepsilon \\ \pm \varepsilon \end{pmatrix} = \begin{pmatrix} \pm 8\varepsilon^2 \\ 1 \\ 1 \pm \varepsilon^2 \end{pmatrix}. \end{aligned}$$

Therefore, $G(X, A)$ is included in the interior of (X, A) if and only if $8\varepsilon^2 < \varepsilon$ or $0 < \varepsilon < 1/8$. This shows that ε cannot be 0 because inclusion in the interior is assumed and it should not be too large to allow (2.3) to be satisfied.

4. Conclusion

A method has been presented allowing the guaranteed inclusion of a solution of the generalized eigenproblem $A - \lambda B$. The corresponding algorithm can be used as an a posteriori criterion to check on the accuracy of computed approximations. Due to a Newton-kind iteration the calculated inclusions are very sharp. The method allows only simple eigenvalues to be treated, in fact the algorithm proves that the enclosed eigenvalue is simple. The problem can easily be transformed into an $n \times n$ nonlinear problem.

The inclusion of multiple eigenvalues is an open problem. All results calculated by the inclusion algorithm are guaranteed to be correct, no false results are possible.

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