BOUNDS FOR THE DETERMINANT BY GERSHGORIN CIRCLES*

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Abstract. Each connected component of the Gershgorin circles of a matrix contains exactly as many eigenvalues as circles are involved. Thus the power set product of all circles is an inclusion of the determinant if all circles are disjoint. We prove that statement to be true for real matrices, even if their circles overlap.

Key words. Gershgorin circle, determinant, power set product.

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The Gershgorin region G of an $n \times n$ complex matrix A contains all its eigenvalues, so there are always n points in G whose product is $\det(A)$. In the case of overlapping Gershgorin circles, one may ask whether these points can be chosen so that one is in each Gershgorin circle of A. We answer that in the affirmative for real A, where in fact all points can be chosen to be real.

THEOREM 1. For a real $n \times n$ matrix A, denote the row sum of absolute values of its off-diagonal elements by $R_k(A) := \sum_{j \neq k} |a_{kj}|$. Then, for $1 \leq k \leq n$, there exist $g_k \in G'_k(A) := \{x \in \mathbb{R} : |a_{kk} - x| \leq R_k(A)\}$ such that

$$\det(A) = \prod_{k=1}^{n} g_k. \tag{1}$$

PROOF. For $n = 1$ there is only one Gershgorin circle and (1) is true. If some diagonal element of A is zero, then the set product of the Gershgorin circles is a disc centered at the origin with radius equal to the product of the ℓ_1-norms of the rows of A. By

$$\{x \in \mathbb{R} : |x| \leq \prod_{k=1}^{n} \|A_{k*}\|_1\} = \{x \in \mathbb{R} : |x| \leq \prod_{k=1}^{n} (|a_{kk}| + R_k(A))\} = \{\prod_{k=1}^{n} g_k : g_k \in G'_k\}$$

the result (1) follows by Hadamard's bound

$$|\det(A)| \leq \prod_{k=1}^{n} \|A_{k*}\|_2 \ . \tag{2}$$

Henceforth we assume without loss of generality that $n \geq 2$ and $a_{ii} \neq 0$ for $1 \leq i \leq n$. Denote the diagonal of A by D, so that $D^{-1}A = I + E$ splits into the identity matrix I and a matrix E with zero diagonal elements. Then $R_k(A) = |d_{kk}| R_k(I + E)$ gives

$$\begin{aligned} G'_k(A) &= d_{kk} \cdot \{d_{kk}^{-1} x \in \mathbb{R} : |a_{kk} - x| \leq R_k(A)\} \\ &= d_{kk} \cdot \{y \in \mathbb{R} : |d_{kk}(1 - y)| \leq R_k(A)\} \\ &= d_{kk} \cdot \{y \in \mathbb{R} : |1 - y| \leq |d_{kk}^{-1}| R_k(A)\} \\ &= d_{kk} \cdot G'_k(I + E). \end{aligned}$$

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Thus, in view of $\det(A) = \det(I+E)\prod_{k=1}^n d_{kk}$, it suffices to prove

$$\forall k \in \{1,\dots,n\}\ \exists g_k \in G'_k(I+E): \quad \det(I+E) = \prod_{k=1}^n g_k \tag{3}$$

for a real matrix E with zero diagonal. Denote by $R_k := R_k(I+E) = R_k(E)$ the absolute row sums of E, and by λ_k the eigenvalues of E. Hadamard's bound (2) implies

$$\det(I+E) \le \prod_{k=1}^n \|(I+E)_{k*}\|_1 = \prod_{k=1}^n (1+R_k) = \max\{\prod_{k=1}^n g_k : g_k \in G'_k(I+E)\}\ ,$$

so that it remains to prove

$$\det(I+E) \ge \min\{\prod_{k=1}^n g_k : g_k \in G'_k(I+E)\}\ . \tag{4}$$

A permutational similarity transformation of $I+E$ neither changes the determinant nor the set of Gershgorin circles. Thus, with suitable ordering of the eigenvalues λ_k, we henceforth assume without loss of generality

$$|\lambda_1| \ge |\lambda_2| \ge \dots \ge |\lambda_n| \qquad \text{and} \qquad R_1 \ge R_2 \ge \dots \ge R_n.$$

We will use three ingredients to prove (4). First, Ostrowski [2] showed

$$\det(I+E) \ge \prod_{k=1}^n (1-R_k) \qquad \text{provided that } R_1 < 1. \tag{5}$$

Second, Hans Schneider [4, Theorem 1] proved

$$\prod_{j=1}^k |\lambda_j| \le \prod_{j=1}^k R_j \qquad \text{for } 1 \le k \le n. \tag{6}$$

Third, Pólya [3] showed that given a real sequence $a_1 \ge a_2 \ge \dots \ge a_n$ and real numbers $b_1, b_2, \dots, b_n$ with

$$\sum_{j=1}^k a_j \le \sum_{j=1}^k b_j \qquad \text{for } 1 \le k \le n, \tag{7}$$

then for any convex increasing function f it follows

$$\sum_{j=1}^k f(a_j) \le \sum_{j=1}^k f(b_j) \qquad \text{for } 1 \le k \le n. \tag{8}$$

If $R_1 < 1$, then (5) shows that

$$\det(I+E) \ge \prod_{k=1}^n (1-R_k) = \min\{\prod_{k=1}^n g_k : g_k \in G'_k(I+E)\}.$$

If $R_1 \geq 1$, we must show that

$$\det(I+E) \geq \min\{\prod_{k=1}^{n} g_k : g_k \in G'_k(E)\} = (1-R_1)\prod_{k=2}^{n}(1+R_k). \tag{9}$$

This is true if $|\lambda_1| \leq 1$ because then $\det(I+E) \geq 0$ by the ordering of the eigenvalues. Otherwise, if $|\lambda_1| > 1$, let k be the greatest index with $\lambda_k \neq 0$. Then (6) implies $R_k > 0$. The sequences $a_j := \log(|\lambda_j|)$ and $b_j := \log(R_j)$ satisfy (7) for $1 \leq j \leq k$ by (6), so that $f(x) := \log(1+e^x)$, (8) and taking the exponential yields[1]

$$\prod_{j=1}^{k}(1+|\lambda_j|) \leq \prod_{j=1}^{k}(1+R_j). \tag{10}$$

Next we show

$$\det(I+E) \geq (1-|\lambda_1|)\prod_{j=2}^{k}(1+|\lambda_j|)\ . \tag{11}$$

Since $|\lambda_1| > 1$ we may restrict our attention to the case $\det(I+E) < 0$, which implies existence of a real eigenvalue $\lambda_m < -1$. Thus

$$\begin{aligned} \det(I+E) &= -|\det(I+E)| = (-|1+\lambda_m|)\textstyle\prod_{j\neq m}|1+\lambda_j| \\ &\geq (-|1+\lambda_m|)\textstyle\prod_{j\neq m}(1+|\lambda_j|) = (1-|\lambda_m|)\prod_{j\neq m}(1+|\lambda_j|). \end{aligned}$$

This proves (11) if $m = 1$, and otherwise we use $|\lambda_1| \geq |\lambda_m|$ and

$$\begin{aligned} (1-|\lambda_m|)(1+|\lambda_1|) &= 1-|\lambda_m|+|\lambda_1|-|\lambda_m\lambda_1| \\ &\geq 1-|\lambda_1|+|\lambda_m|-|\lambda_1\lambda_m| = (1-|\lambda_1|)(1+|\lambda_m|). \end{aligned}$$

Hence (11), (10) and $|\lambda_1| \leq R_1$ give

$$\begin{aligned} \det(I+E) &\geq (1-|\lambda_1|)\textstyle\prod_{j=2}^{k}(1+|\lambda_j|) \geq \dfrac{1-|\lambda_1|}{1+|\lambda_1|}\displaystyle\prod_{j=1}^{k}(1+R_j) \\ &\geq (1-R_1)\textstyle\prod_{j=2}^{k}(1+R_j) \geq (1-R_1)\prod_{j=2}^{n}(1+R_j), \end{aligned}$$

which is (9) and completes the proof. □

Given an approximate LU-decomposition of a real matrix A. let $X \approx L^{-1}$ and $Y \approx U^{-1}$ be approximate inverses with unit lower triangular X. Then $\det(A) = \det(XAY)/\prod Y_{kk}$, where we can expect that $M := XAY$ is a small perturbation of the identity matrix. To compute an inclusion of $\det(M)$ the interval product of the Gershgorin circles of M was used in the Reliable Computing community. Theorem 1 justifies that approach.

We conclude with the question whether for complex A there exist complex points g_k in each Gershgorin circle G_k such that $\prod_{k=1}^{n} g_k = \det(A)$.

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[1] As noted by the referee, (10) is known. It follows by Exercise (3.3.23) in [1] and replacing singular values by R_j as in Weyl's product inequalities.

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