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VARIATIONAL CHARACTERIZATIONS OF THE SIGN-REAL AND THE SIGN-COMPLEX SPECTRAL RADIUS

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Abstract. The sign-real and the sign-complex spectral radius, also called the generalized spectral radius, proved to be an interesting generalization of the Perron-Frobenius theory for nonnegative matrices to general real and to general complex matrices, respectively. Especially the generalization of the well-known Collatz-Wielandt max-min characterization shows one of the many one-to-one correspondences to classical Perron-Frobenius theory. In this paper we prove variational characterizations of the generalized (real and complex) spectral radius which are again almost identical to the corresponding one in classical Perron-Frobenius theory.

1. Introduction. Denote $\mathbb{R}_+ := \{x \geq 0 : \ x \in \mathbb{R}\}$, and let $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbf{C}\}$. The generalized spectral radius is defined by

$$(1) \qquad \rho^{\mathbb{K}}(A) := \max\{|\lambda| : \ \exists\, 0 \neq x \in \mathbb{K}^n, \ \exists\, \lambda \in \mathbb{K}, \ |Ax| = |\lambda x|\} \qquad \text{for } A \in M_n(\mathbb{K}).$$

Note that absolute value and comparison of matrices and vectors are always to be understood componentwise. For example, $A \leq |C|$ for $A \in M_n(\mathbb{R})$, $C \in M_n(\mathbf{C})$ is equivalent to $A_{ij} \leq |C_{ij}|$ for all i, j.

For $\mathbb{K} = \mathbb{R}_+$ the quantity in (1.1) is the classical Perron root, for $\mathbb{K} \in \{\mathbb{R}, \mathbf{C}\}$ it is the sign-real or sign-complex spectral radius, respectively. Note that the quantities are only defined for matrices out of the specific set $\mathbb{K}$, and note that for $\rho^{\mathbb{R}}$ the maximum is taken over $|\lambda|$ for $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$. Vectors $0 \neq x \in \mathbb{K}^n$ and scalars $\lambda \in \mathbb{K}$ satisfying the nonlinear eigenequation $|Ax| = |\lambda x|$ are also called generalized eigenvectors and generalized eigenvalues, respectively.

Denote the set of signature matrices over $\mathbb{K}$ by $\mathcal{S}(\mathbb{K})$, which are diagonal matrices S with $|S_{ii}| = 1$ for all i. In short notation $S \in \mathcal{S}(\mathbb{K}) :\Leftrightarrow \ S \in M_n(\mathbb{K})$ and $|S| = I$. For $\mathbb{K} = \mathbb{R}_+$ this is just the identity matrix I, for $\mathbb{K} = \mathbb{R}$ the set of diagonal orthogonal or $S = \operatorname{diag}(\pm 1)$, and for $\mathbb{K} = \mathbf{C}$ the set of diagonal unitary matrices. Obviously, for $y \in \mathbb{K}$ there is $S \in \mathcal{S}(\mathbb{K})$ with $Sy \geq 0$. In case $|y| > 0$, this S is uniquely determined. Note that $S^{-1} = S^* \in \mathcal{S}(\mathbb{K})$ for all $S \in \mathcal{S}(\mathbb{K})$.

By definition (1.1) there is $y \in \mathbb{K}^n$ with $|Ay| = |ry| = r|y|$ for $r := \rho^{\mathbb{K}}(A)$, and therefore for $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbf{C}\}$,

$$(2) \qquad \exists\, S \in \mathcal{S}(\mathbb{K}) \ \exists\, 0 \neq y \in \mathbb{K}^n : \ SAy = ry$$

and

$$(3) \qquad \exists\, S_1, S_2 \in \mathcal{S}(\mathbb{K}) \ \exists\, x \geq 0 : \ S_1 A S_2 x = rx.$$

Among the variational characterizations of the Perron root are

$$(4) \qquad \max_{x \geq 0} \min_{x_i \neq 0} \frac{(Ax)_i}{x_i} = \rho^{\mathbb{R}_+}(A) = \rho(A) = \inf_{x > 0} \max_i \frac{(Ax)_i}{x_i} \quad \text{for} \quad A \geq 0$$

and

$$(5) \qquad \max_{x \geq 0} \min_{\substack{y \geq 0 \\ y^T x \neq 0}} \frac{y^T A x}{y^T x} = \rho(A) = \min_{y \geq 0} \max_{\substack{x \geq 0 \\ y^T x \neq 0}} \frac{y^T A x}{y^T x} \text{ for } A \geq 0.$$

The purpose of this paper is to prove a generalization of both characterizations for the generalized spectral radius.

We note that the only non-obvious property of the generalized spectral radius we use is [6, Corollary 2.4]

$$(6) \qquad \rho^{\mathbb{K}}(A[\mu]) \leq \rho^{\mathbb{K}}(A) \quad \text{for } \mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbf{C}\}, \ A \in M_n(\mathbb{K}) \text{ and } \mu \subseteq \{1, \ldots, n\}.$$

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2. Variational characterizations. For the following results we need two preparatory lemmata, the first showing that there exists a generalized eigenvector in every orthant.

LEMMA 2.1. *Let* $\mathbb{K} \in \{\mathbb{R}, \mathbf{C}\}$ *and* $A \in M_n(\mathbb{K})$ *be given. Then*

$$\forall\, S \in \mathcal{S}(\mathbb{K})\ \exists\, 0 \neq z \in \mathbb{K}^n\ \exists\, \lambda \in \mathbb{R}_+ : \quad Sz \geq 0,\ |Az| = \lambda |z|.$$

Remark. The condition $Sz \geq 0$ for $z \in \mathbb{K}^n$ means $Sz \in \mathbb{R}^n$ and $Sz \geq 0$, or shortly $Sz \in \mathbb{R}^n_+$. Note that Lemma 2.2 is also true for $\mathbb{K} = \mathbb{R}_+$, in which case $S \in \mathcal{S}(\mathbb{K})$ implies $S = I$.

Proof. Let $S \in \mathcal{S}(\mathbb{K})$ be given and define $\mathcal{O} := \{z \in \mathbb{K}^n :\ \|z\|_1 = 1, Sz \geq 0\}$. The set $\mathcal{O}$ is nonempty, compact and convex. If there exists some $z \in \mathcal{O}$ with $Az = 0$ we are finished with $\lambda = 0$. Suppose $Az \neq 0$ for all $z \in \mathcal{O}$ and define $\varphi(x) := \|Ax\|_1^{-1} \cdot S^*|Ax|$. Then φ is well-defined on $\mathcal{O}$ and $\varphi :\ \mathcal{O} \to \mathcal{O}$, such that by Brouwer's theorem there exists a fixed point $z \in \mathcal{O}$ with $\varphi(z) = \|Az\|_1^{-1} \cdot S^*|Az| = z$. Then $|Az| = \lambda Sz = \lambda |z|$ with $\lambda = \|Az\|_1$. ∎

The next lemma states a property of vectors out of the interior of a certain orthant.

LEMMA 2.2. *Let* $\mathbb{K} \in \{\mathbb{R}, \mathbf{C}\}$*,* $A \in M_n(\mathbb{K})$ *and define* $r := \rho^{\mathbb{K}}(A)$*. Then*

$$\forall\, S \in \mathcal{S}(\mathbb{K})\ \forall\, \varepsilon > 0\ \exists\, z \in \mathbb{K}^n : \quad Sz > 0,\ |Az| \leq (r + \varepsilon) \cdot |z|.$$

Proof. We proceed by induction. For $n = 1$, it is $r = |A_{11}| \in \mathbb{R}_+$, and $z := \operatorname{sign}(S_{11}) \in \mathbb{K}$ does the job. Suppose the lemma is proved for dimension less than n. For given $S \in \mathcal{S}(\mathbb{K})$ there exists by Lemma 2.1 some $0 \neq z \in \mathbb{K}^n$ and $\lambda \in \mathbb{R}_+$ with $Sz \geq 0$ and $|Az| = \lambda |z|$. If $Sz > 0$ we are finished. Denote $\mu := \{j :\ z_j \neq 0\}$ and set $\nu := \{1, \ldots, n\} \backslash \mu$, such that

$$\left| \begin{pmatrix} T & U \\ V & W \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} \right| = \lambda \left| \begin{pmatrix} x \\ 0 \end{pmatrix} \right| \text{ with} \tag{7}$$
$$T = A[\mu],\ U = A[\mu, \nu],\ V = A[\nu, \mu],\ W = A[\nu],\ z[\mu] = x \text{ and } z[\nu] = 0.$$

Then $|Tx| = \lambda |x|$, $Vx = 0$ and $|x| > 0$.

By induction hypothesis there exists $y' \in \mathbb{K}^{|\nu|}$ with $S[\nu]y' > 0$ and

$$|Wy'| \leq (\rho^{\mathbb{K}}(W) + \varepsilon)|y'| \leq (r + \varepsilon)|y'|,$$

where the latter inequality follows by (1.6). Define

$$\alpha := \begin{cases} \min\limits_i \left| \dfrac{x_i}{(Uy')_i} \right| & \text{for } (Uy')_i \neq 0 \\ 1 & \text{otherwise,} \end{cases}$$

and set $y := \alpha y'$. Then $|y| > 0$ and

$$\left| A \cdot \begin{pmatrix} x \\ \varepsilon y \end{pmatrix} \right| = \left| \begin{pmatrix} Tx + \varepsilon Uy \\ \varepsilon Wy \end{pmatrix} \right| \leq \begin{pmatrix} \lambda |x| + \varepsilon\alpha |Uy'| \\ \varepsilon\alpha (r + \varepsilon)|y'| \end{pmatrix} = (r + \varepsilon) \begin{pmatrix} |x| \\ \varepsilon |y| \end{pmatrix}.$$

∎

The above lemma is obviously not true when replacing $r + \varepsilon$ by r, as the example $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with $\rho^{\mathbb{K}}(A) = 1$ for $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbf{C}\}$ shows. It is, at least for $\mathbb{K} = \mathbb{R}$, also not valid when $|A|$ is irreducible. Consider

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

It has been shown in [5, Lemma 5.6] that $\rho^{\mathbb{R}}(A) = 1$. We show that $|Au| \le u$ is not possible for $u > 0$. Set $u := (x, y, z)^T$, then $|Au| \le u$ is equivalent to

$$\begin{array}{rcccl} -x & \le & y+z & \le & x \\ -y & \le & -x+z & \le & y \\ -z & \le & -x-y & \le & z. \end{array}$$

The second and third row imply

$$x \le y + z \quad \text{and} \quad y \le -x + z,$$

and by the first and second row,

$$x = y + z \quad \text{and} \quad y = -x + z$$

so that $y = x - z = -x + z$ and therefore $y = 0$, which means u cannot be positive.

Finally, we need a generalization of a theorem by Collatz [3, Section 2] to the complex case.

LEMMA 2.3. *Let $A \in M_n(\mathbf{C})$, $A^* z = \lambda z$ for $0 \ne z \in \mathbb{R}^n$, $\lambda \in \mathbf{C}$. Then for all $x \in \mathbb{R}^n$ with $|x| > 0$ and $x_i z_i \ge 0$ for all i the following estimations hold true:*

$$\begin{array}{rcccl} \min \operatorname{Re} \mu_i & \le & \operatorname{Re} \lambda & \le & \max \operatorname{Re} \mu_i \\ \min \operatorname{Im} \mu_i & \le & \operatorname{Im} \lambda & \le & \max \operatorname{Im} \mu_i, \end{array}$$

where $\mu_i := (Ax)_i / x_i$ for $1 \le i \le n$.

Remark. Note that x and the left eigenvector z are assumed to be real.

Proof. Similar to Collatz's original proof for the case $A \ge 0$ we note

$$\sum_i (\lambda - \mu_i) x_i z_i = \sum_i x_i (A^* z)_i - \sum_i (Ax)_i z_i = x^T A^* z - z^T A x = 0.$$

Now $x_i z_i$ are real nonnegative for all i, and by $|x| > 0$ not all products $x_i z_i$ can be zero. The assertion follows. ∎

With these preparations we can prove the first two-sided characterization of $\rho^{\mathbb{K}}$.

THEOREM 2.4. *Let $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbf{C}\}$ and $A \in M_n(\mathbb{K})$. Then*

$$\max_{S \in \mathcal{S}(\mathbb{K})} \max_{\substack{x \in \mathbb{K}^n \\ Sx \ge 0}} \min_{x_i \ne 0} \left| \frac{(Ax)_i}{x_i} \right| = \rho^{\mathbb{K}}(A) = \max_{S \in \mathcal{S}(\mathbb{K})} \inf_{\substack{x \in \mathbb{K}^n \\ Sx > 0}} \max_i \left| \frac{(Ax)_i}{x_i} \right|. \tag{8}$$

Remark. The characterization is almost identical to the classical Perron-Frobenius characterization (1.4). The difference is that for nonnegative A the nonnegative orthant is the generic one, and vectors x can be restricted to this generic orthant. For general real or complex matrices, there is no longer a generic orthant, and henceforth the max-min and inf-max characterization is maximized over all orthants. Note that in the left hand side the two maximums can be replaced by $\max_{x \in \mathbb{K}^n}$, but are separated for didactical purposes.

Proof. The result is well known for $\mathbb{K} = \mathbb{R}_+$, and the left equality was shown in [5, Theorem 3.1] for $\mathbb{K} = \mathbb{R}$, and for $\mathbb{K} = \mathbf{C}$ it was shown in a different context in [4] and [2], see also [6, Theorem 2.3]. We need to prove the right equality for $\mathbb{K} \in \{\mathbb{R}, \mathbf{C}\}$. Let $S \in \mathcal{S}(\mathbb{K})$ be fixed but arbitrary and denote $r := \rho^{\mathbb{K}}(A)$. By Lemma 2.2, there exists for every $\varepsilon > 0$ some $x \in \mathbb{K}^n$ with $Sx > 0$ and $|Ax| \le (r + \varepsilon)|x|$, so that $r \ge$ r.h.s.(2.2). We will prove $r \le$ r.h.s.(2.2) to finish the proof. By (1.3) and $\rho^{\mathbb{K}}(A^*) = \rho^{\mathbb{K}}(A)$ there is $S_1, S_2 \in \mathcal{S}(\mathbb{K})$ and $0 \ne z \in \mathbb{R}^n$ with $z \ge 0$ and $S_1 A^* S_2 z = rz$. Then for any $x \in \mathbb{K}^n$ with $S_1 x > 0$, Lemma 2.3 implies

$$\max_i \left| \frac{(Ax)_i}{x_i} \right| = \max_i \left| \frac{((S_2^* A S_1^*) \cdot S_1 x)_i}{(S_1 x)_i} \right| \ge \operatorname{Re} r = r. \qquad ∎$$

Next we give a second two-sided characterization of the generalized spectral radius.

THEOREM 2.5. *Let* $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbf{C}\}$ *and* $A \in M_n(\mathbb{K})$. *Then*

$$\max_{S_1,S_2\in\mathcal{S}(\mathbb{K})} \max_{\substack{x\in\mathbb{K}^n \\ S_1x\geq 0}} \min_{\substack{y\in\mathbb{K}^n \\ S_2y\geq 0 \\ |y^*|\,|x|\neq 0}} \frac{|y^*Ax|}{|y^*|\,|x|} = \rho^{\mathbb{K}}(A) = \max_{S_1,S_2\in\mathcal{S}(\mathbb{K})} \min_{\substack{y\in\mathbb{K}^n \\ S_2y\geq 0}} \max_{\substack{x\in\mathbb{K}^n \\ S_1x\geq 0 \\ |y^*|\,|x|\neq 0}} \frac{|y^*Ax|}{|y^*|\,|x|}. \tag{9}$$

Proof. Let, according to (1.2), $SAx = rx$ for $S \in \mathcal{S}(\mathbb{K})$, $0 \neq x \in \mathbb{K}^n$ and $r = \rho^{\mathbb{K}}(A)$. Define S_1 such that $S_1x \geq 0$ and set $S_2 = S_1S$. Then for every $y \in \mathbb{K}^n$ with $S_2y \geq 0$ and $|y^*|\,|x| \neq 0$, it is $S_1x = |x|$, $S_2y = |y|$, $S_2^*S_1S = I$ and

$$y^*Ax = y^*S_2^*S_1SAx = ry^*S_2^*S_1x = r|y^*|\,|x|, \quad \text{or} \quad |y^*Ax| = r|y^*|\,|x|.$$

That means for the specific choice of S_1, S_2 and x, the ratio $\frac{|y^*Ax|}{|y^*|\,|x|}$ is equal to r independent of the choice of y provided $S_2y \geq 0$. Therefore, both the left and the right hand side of (2.3) are greater than or equal to $\rho^{\mathbb{K}}(A)$. This proves also that the extrema are actually achieved.

On the other hand, let $S_1, S_2 \in \mathcal{S}(\mathbb{K})$ and $x \in \mathbb{K}^n$, $S_1x \geq 0$ be fixed but arbitrarily given. Denote $\mu := \{j : x_j \neq 0\}$, $k := |\mu|$, and $\overline{\mu} := \{1, \ldots, n\}\backslash\mu$. By Lemma 2.1, there exists $\widetilde{y} \in \mathbb{K}^k$ with $\widetilde{y} \neq 0$, $S_2[\mu]\,\widetilde{y} \geq 0$ and $|A^*[\mu] \cdot \widetilde{y}\,| = \lambda|\,\widetilde{y}\,|$ for $\lambda \geq 0$. Define $y \in \mathbb{K}^n$ by $y[\mu] := \widetilde{y}$ and $y[\overline{\mu}] := 0$. Then $x[\overline{\mu}] = 0$ implies $|y^*|\,|x| = |y[\mu]^*|\,|x[\mu]|$ and

$$|y^*Ax| = |y[\mu]^*A[\mu]x[\mu]| \leq |y[\mu]^*A[\mu]| \cdot |x[\mu]| = \lambda|y[\mu]^*|\,|x[\mu]| = \lambda|y^*|\,|x|.$$

By (1.6),

$$\frac{|y^*Ax|}{|y^*|\,|x|} \leq \lambda \leq \rho^{\mathbb{K}}(A).$$

Henceforth, for that choice of y (depending on S_1, S_2 and x) the left hand side of (2.3) is less than or equal to $\rho^{\mathbb{K}}(A)$. It remains to prove that the right hand side of (2.3) is less than or equal to $\rho^{\mathbb{K}}(A)$. Let S_1, S_2 be given, fixed but arbitrary. By Lemma 2.1, there exists $0 \neq y \in \mathbb{K}^n$ with $S_2y \geq 0$ and $|A^*y| = \lambda|y|$ for $\lambda \in \mathbb{R}_+$. Then for all $x \in \mathbb{K}^n$,

$$|y^*Ax| \leq |y^*A|\,|x| = \lambda|y^*|\,|x|,$$

such that for that choice of y (depending on S_1, S_2) the ratio $\frac{|y^*Ax|}{|y^*|\,|x|}$ is equal to λ for all $x \in \mathbb{K}^n$ with $|y^*|\,|x| \neq 0$. It follows that the right hand side of (2.3) is less than or equal to $\lambda \leq \rho^{\mathbb{K}}(A^*) = \rho^{\mathbb{K}}(A)$, and the proof is finished. ■

We note that Theorem 2.5 and its proof cover the case $\mathbb{K} = \mathbb{R}_+$, where in this case $\mathcal{S}(\mathbb{R}_+)$ consists only of the identity matrix.

Finally we notice that for the classical Perron-Frobenius theory the characterization (2.3) is mentioned without proof in the classical book by Varga [7] for irreducible matrices. As in other text books, the result is referenced to be included in [1], where in turn we only found a reference to an internal report.

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