Complex Disk Products and Cartesian Ovals

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Abstract. Let D_R, D_r, D_S, D_s be complex disks with common center 1 and radii R, r, S, s, respectively. We consider the Minkowski products $A := D_R D_r$ and $B := D_S D_s$ and give necessary and sufficient conditions for A being a subset or superset of B. Partially, this extends to n-fold disk products $D_1 \cdots D_n$, $n > 2$.

It is well-known that the boundaries of A and B are outer loops of Cartesian ovals. Therefore, our results translate to necessary and sufficient conditions under which such loops encircle each other.

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Denote by

$$D(c, \delta) := \{z \in \mathbb{C} : |z - c| \leq \delta\}$$

the closed complex disk with center $c \in \mathbb{C}$ and radius $\delta \in \mathbb{R}_{\geq 0}$. For given disks D_1, D_2 let D_1', D_2' be disks with the same product of centers but different radii. We will give necessary and sufficient conditions for the fact that the Minkowski product $D_1 D_2$ is a subset or superset of $D_1' D_2'$.

Recently it was proved [3] that the determinant of a real matrix is contained in the Minkowski product of its Gershgorin circles. This is nontrivial if the Gershgorin circles overlap; the proof uses only results from classical matrix theory. When extending this result to complex matrices [4] we arrived at the problem addressed in this note.

Independent from that the problem is of general interest within the context of the Minkowski algebra of complex sets. This set algebra is thoroughly described by Farouki, Moon, and Ravani in [1] along with various applications, and we refer to that paper and the references therein for details on that topic. Let us particularly mention that [1] stresses the connection between complex circle products, cartesian ovals, and geometrical optics to demonstrate practical relevance.

Let $D_1 = D(c_1, \delta_1)$ and $D_2 = D(c_2, \delta_2)$ be given. If $c_1 c_2 = 0$, then the Minkowski product $D_1 D_2$ is simply a circle around the origin with radius

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$(|c_1|+\delta_1)(|c_2|+\delta_2)$, and the desired inclusion conditions follow immediately. If $c_1c_2 \neq 0$, then

$$D_1D_2 = c_1c_2D(1,r_1)D(1,r_2) \quad \text{with } r_1 = \frac{\delta_1}{|c_1|} \text{ and } r_2 = \frac{\delta_2}{|c_2|}.$$

Therefore, we may restrict our attention to products of disks with center 1.

Denote by $A := D(1,R)\cdot D(1,r)$ and $B := D(1,S)\cdot D(1,s)$ the Minkowski products of two pairs of disks with common center 1 and radii $R, r, S, s \in \mathbb{R}_{\geq 0}$. It is well-known that the boundary of such a Minkowski product is the outer loop of a Cartesian oval [1, 2]. The extremal real points of A are well-known to be

$$\begin{aligned} \lambda_A &:= \min A \cap \mathbb{R} = \min\,(1-R)(1\pm r) = 1-R-r|1-R| && (1)\\ \rho_A &:= \max A \cap \mathbb{R} = (1+R)(1+r) && (2) \end{aligned}$$

with similar formulas for B. It follows that

$$[\lambda_A, \rho_A] = A\cap\mathbb{R} \qquad \text{and} \qquad [\lambda_B, \rho_B] = B\cap\mathbb{R}.$$

The subject of this note is to present necessary and sufficient conditions for A being a subset or superset of B. This is a question of matter if $R \geq S \geq s \geq r$. Our main result is

Theorem 1. *Let $R \geq S \geq s \geq r \in \mathbb{R}_{\geq 0}$ be given. Define $A := D(1,R)\cdot D(1,r)$, $B := D(1,S)\cdot D(1,s)$. Then, the following equivalence relations hold true:*

$$\begin{array}{lllllll} \text{a)} & A \subseteq B & \Leftrightarrow & \lambda_A \in B & \Leftrightarrow & \lambda_B \leq \lambda_A \\ \text{b)} & B \subseteq A & \Leftrightarrow & \rho_B \in A & \Leftrightarrow & \rho_B \leq \rho_A \end{array}$$

This means that the whole set A is contained in B if, and only if, its left-most real point $\lambda_A = 1-R-r|1-R|$ is an element of B. Likewise, the set B is contained in A if, and only if, its right-most real point $\rho_B = (1+S)(1+s)$ is an element of A.

Inclusions in the respective topological interiors are characterized as follows.

Corollary 2. *With the notation of Theorem 1 the following equivalence relations hold true:*

$$\begin{array}{lllllll} \text{a)} & A \subseteq \overset{\circ}{B} & \Leftrightarrow & \lambda_A \in \overset{\circ}{B} & \Leftrightarrow & \lambda_B < \lambda_A \\ \text{b)} & B \subseteq \overset{\circ}{A} & \Leftrightarrow & \rho_B \in \overset{\circ}{A} & \Leftrightarrow & \rho_B < \rho_A \end{array}$$

where $\overset{\circ}{M}$ denotes the interior of a set M.

We start with proving relations between the extremal real points of A and B.

Lemma 3. *Let $R \geq S \geq s \geq r \in \mathbb{R}_{\geq 0}$ and define*

$$\begin{array}{llll} \lambda_A & := 1-R-r|1-R|, & \rho_A & := 1+R+r+Rr, \\ \lambda_B & := 1-S-s|1-S|, & \rho_B & := 1+S+s+Ss. \end{array}$$

Then

a) $\quad \lambda_B \leq \lambda_A \quad \Rightarrow \quad \rho_A \leq \rho_B$

b) $\quad \rho_A \geq \rho_B \quad \Rightarrow \quad \lambda_A \leq \lambda_B$

Proof. a) Suppose that $\lambda_B \leq \lambda_A$.

Case 1: $S \geq 1$. Then, $R \geq 1$ and using $r \leq s$ gives

$$\begin{aligned}\rho_A - 1 &= R - r + Rr + 2r = 1 - \lambda_A + 2r \leq 1 - \lambda_B + 2s \\ &= S - s + Ss + 2s = \rho_B - 1.\end{aligned}$$

Case 2: $S < 1$. Then $\lambda_B = (1-S)(1-s) > 0$ so that $R \geq 1$ is not possible because otherwise $\lambda_A = (1-R)(1+r) \leq 0 < \lambda_B$. Thus, $R < 1$ and $\lambda_A = (1-R)(1-r)$. Abbreviate

$$\mu := \frac{R+r}{2},\ \Delta := \frac{R-r}{2}, \quad \text{and} \quad \tilde{\mu} := \frac{S+s}{2},\ \tilde{\Delta} := \frac{S-s}{2}. \tag{3}$$

Then $\Delta \geq \tilde{\Delta}$, and $0 \leq \mu, \tilde{\mu} \leq 1$ together with

$$(1-\tilde{\mu})^2 - \Delta^2 \leq (1-\tilde{\mu})^2 - \tilde{\Delta}^2 = (1-S)(1-s) = \lambda_B \leq \lambda_A = (1-\mu)^2 - \Delta^2$$

imply $\mu \leq \tilde{\mu}$. Hence

$$\rho_A = (1+\mu)^2 - \Delta^2 \leq (1+\tilde{\mu})^2 - \tilde{\Delta}^2 = \rho_B.$$

b) Suppose that $\rho_A \geq \rho_B$.

Case 1: $S \geq 1$. Using $R \geq S \geq 1$ and $r \leq s$ gives

$$\begin{aligned}1 - \lambda_A &= R + r(R-1) = \rho_A - 1 - 2r \geq \rho_B - 1 - 2s = S + s + Ss - 2s \\ &= 1 - \lambda_B.\end{aligned}$$

Case 2: $S < 1$. Then $\lambda_B = (1-S)(1-s) > 0$. If $R \geq 1$, then $\lambda_A = (1-R)(1+r) \leq 0 < \lambda_B$. Thus, we may assume that $R < 1$. We use (3) again, so that $\Delta \geq \tilde{\Delta}$ and

$$(1+\mu)^2 - \tilde{\Delta}^2 \geq (1+\mu)^2 - \Delta^2 = \rho_A \geq \rho_B = (1+\tilde{\mu})^2 - \tilde{\Delta}^2$$

imply $\mu \geq \tilde{\mu}$. Thus, using $0 \leq \mu, \tilde{\mu} \leq 1$,

$$\lambda_A = (1-\mu)^2 - \Delta^2 \leq (1-\tilde{\mu})^2 - \tilde{\Delta}^2 = \lambda_B$$

finishes that case and the proof. $\square$

In [4] the following parameterization of the outer loop of a Cartesian oval was invented.

Lemma 4. *Let $R \geq r \in \mathbb{R}_{\geq 0}$ be given. The boundary of the Minkowski (set) product $A = D(1,R) \cdot D(1,r)$ is parameterized by $z(t) = x(t) + iy(t)$ with*

$$x(t) := \frac{t^2 + 1 - (R^2 + r^2)}{2} \tag{4}$$

$$y(t) := \pm\sqrt{(R+tr)^2 - (x(t) - 1 + r^2)^2} \tag{5}$$

$$t \in [t_1, t_2] := [\ |1-R| - r\ ,\ 1 + R + r\ ]. \tag{6}$$

Note that the extremal real points of A satisfy $\lambda_A = x(t_1)$ and $\rho_A = x(t_2)$.

Next, we prove Theorem 1, that is, necessary and sufficient conditions for $A \subseteq B$ or for $B \subseteq A$. Two typical cases are displayed in Figure 1, where A is depicted by the black solid and B by the red dashed line. We will show that $A \subseteq B$ holds true if, and only if, $\lambda_A \in B$, and $B \subseteq A$ if, and only if, $\rho_B \in A$.

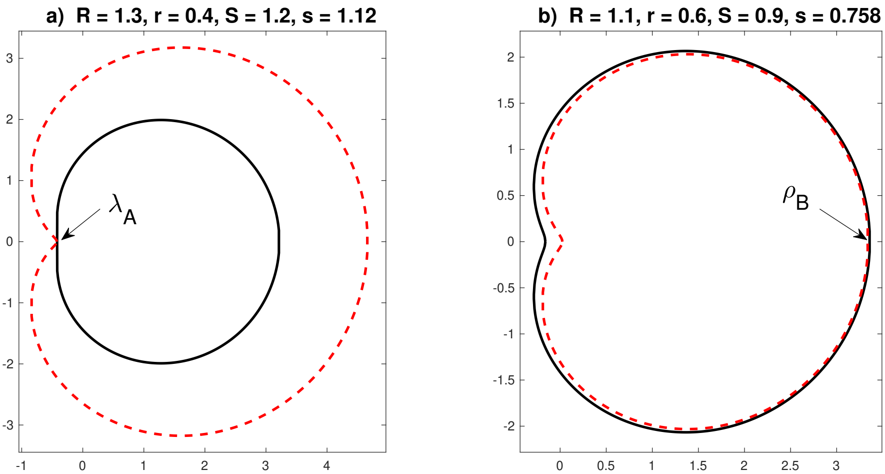

FIGURE 1. Outer loops of Cartesian ovals for products of complex circles with center 1.

Proof of Theorem 1. The implications

$$\begin{array}{ccccc} A \subseteq B & \Rightarrow & \lambda_A \in B & \Rightarrow & \lambda_B \le \lambda_A \\ B \subseteq A & \Rightarrow & \rho_B \in A & \Rightarrow & \rho_B \le \rho_A \end{array}$$

are trivial which leaves us with proving the following two implications:

$$\text{(i)} \quad \lambda_B \le \lambda_A \;\Rightarrow\; A \subseteq B \qquad\qquad \text{(ii)} \quad \rho_B \le \rho_A \;\Rightarrow\; B \subseteq A$$

By Lemma 4 the outer loop of A has the parameterization (4)-(6), and analogously, $\tilde{x}, \tilde{y}, \tilde{t}_1, \tilde{t}_2$ are defined for B by replacing R, r by S, s. A computation gives

$$x(t)^2 + y(t)^2 \;=\; (t + Rr)^2 \quad \text{for all } t \in [t_1, t_2], \tag{7}$$

$$\tilde{x}(t)^2 + \tilde{y}(t)^2 \;=\; (t + Ss)^2 \quad \text{for all } t \in [\tilde{t}_1, \tilde{t}_2]. \tag{8}$$

Since A and B are symmetric to the real axis we may restrict our considerations to the closed upper complex half plane where A and B have the respective boundary curves $z(t) := x(t) + iy(t)$ and $\tilde{z}(t) := \tilde{x}(t) + i\tilde{y}(t)$. These curves connect λ_A, ρ_A and λ_B, ρ_B, respectively. Precisely,

$$[z(t_1), z(t_2)] = [\lambda_A, \rho_A] \qquad \text{and} \qquad [\lambda_B, \rho_B] = [\tilde{z}(\tilde{t}_1), \tilde{z}(\tilde{t}_2)]. \tag{9}$$

ad (i). By Lemma 3 a) and (9),

$$A \cap \mathbb{R} = [z(t_1), z(t_2)] = [\lambda_A, \rho_A] \subseteq [\lambda_B, \rho_B] = [\tilde{z}(\tilde{t}_1), \tilde{z}(\tilde{t}_2)] = B \cap \mathbb{R}. \quad (10)$$

Thus, the endpoints of the boundary curve $z(t)$ of A are in B and the assertion follows if $z(t)$ does not leave B. In order to derive a contradiction, we assume that there is a point $\hat{z} = z(\hat{t}) \in A \backslash B$. Then, injectivity of the curves $z(t)$ and $\tilde{z}(t)$ implies that they must have (at least) two distinct intersection points $z_k := z(\tau_k) = \tilde{z}(\tilde{\tau}_k)$, $\tau_k \in [t_1, t_2]$ and $\tilde{\tau}_k \in [\tilde{t}_1, \tilde{t}_2]$, $k = 1, 2$. Now, $x(\tau_k) = \tilde{x}(\tilde{\tau}_k)$ and (4) yield

$$\tilde{\tau}_k^2 - \tau_k^2 \quad = \quad S^2 + s^2 - (R^2 + r^2). \quad (11)$$

From (7), (8) it follows that $(\tau_k + Rr)^2 = (\tilde{\tau}_k + Ss)^2$, i.e., $\tilde{\tau}_k^2 = (\tau_k + Rr + \alpha Ss)^2$ for some $\alpha \in \{-1, 1\}$. Therefore, (11) becomes

$$2(Rr + \alpha Ss)\tau_k + (Rr + \alpha Ss)^2 \quad = \quad S^2 + s^2 - (R^2 + r^2). \quad (12)$$

If $Rr + \alpha Ss = 0$, then $Rr = Ss$ and (12) transforms to

$$0 \quad = \quad S^2 + s^2 - (R^2 + r^2) = (S - s)^2 - (R - r)^2.$$

which implies $(S, s) = (R, r)$ so that $A = B$. If $Rr + \alpha Ss \neq 0$, then (12) gives

$$\tau_k \quad = \quad \frac{S^2 + s^2 - (R^2 + r^2) - (Rr + \alpha S)^2}{2(Rr + \alpha S)}.$$

Since the right-hand side is independent of k we get $\tau_1 = \tau_2$ and therefore $z_1 = z(\tau_1) = z(\tau_2) = z_2$, a contradiction as z_1 and z_2 are distinct.

ad (ii). Lemma 3 b) and (9) yield

$$B \cap \mathbb{R} = [\tilde{z}(\tilde{t}_1), \tilde{z}(\tilde{t}_2)] = [\lambda_B, \rho_B] \subseteq [\lambda_A, \rho_A] = [z(t_1), z(t_2)] = A \cap \mathbb{R}.$$

Now, the proof proceeds exactly the same way as in a) with the roles of A and B interchanged. □

Proof of Corollary 2. The implications

$$\begin{array}{ccccc} A \subseteq \overset{\circ}{B} & \Rightarrow & \lambda_A \in \overset{\circ}{B} & \Rightarrow & \lambda_B < \lambda_A \\ B \subseteq \overset{\circ}{A} & \Rightarrow & \rho_B \in \overset{\circ}{A} & \Rightarrow & \rho_B < \rho_A \end{array}$$

are again trivial and we are left with proving:

$$\text{(i)} \quad \lambda_B < \lambda_A \;\Rightarrow\; A \subseteq \overset{\circ}{B} \qquad\qquad \text{(ii)} \quad \rho_B < \rho_A \;\Rightarrow\; B \subseteq \overset{\circ}{A}$$

ad (i). We may assume that $r < s$ since otherwise $r = s$ clearly implies $B \subseteq A$ and therefore $\lambda_A \leq \lambda_B$. By continuity we can find $\tilde{s} \in (r, s)$ such that $\lambda_B < \lambda_{\tilde{B}} < \lambda_A$ for $\tilde{B} := D(1, S)D(1, \tilde{s})$. By Theorem 1 a), $A \subseteq \tilde{B}$. Furthermore,[1]

$$\tilde{B} \subseteq D(1, S)\overset{\circ}{D}(1, s) = \{z\overset{\circ}{D}(1, s) : z \in D(1, S)\},$$

[1]Here $\overset{\circ}{D}(1, s)$ denotes the interior of $D(1, s)$ which is the open disk with center 1 and radius s.

which is a union of open sets, so that $A \subseteq \tilde{B} \subseteq \mathring{B}$.

ad (ii). We may assume that $R > S$ because $R = S$ implies

$$\rho_A = (1+R)(1+r) \le (1+S)(1+s) = \rho_B.$$

We proceed analogously to (i) choosing $\tilde{R} \in (S, R)$. $\square$

Keeping the notation of Theorem 1 and Corollary 2, the special case $Rr = Ss$ was considered in [4], Lemma 3, where $B \subseteq A$ was deduced more directly. This result follows in a sharpened form from the next Corollary.

Corollary 5. *If $(R, r) \ne (S, s)$ and $Rr = Ss$, then $B \subseteq \mathring{A}$.*

Proof. From $R \ge S \ge s \ge r$ and $(R, r) \ne (S, s)$ we have $\sqrt{R} - \sqrt{r} > \sqrt{S} - \sqrt{s}$. Abbreviating $c := Rr = Ss$, it follows that

$$\begin{aligned} \rho_A - 1 &= R + r + Rr = (\sqrt{R} - \sqrt{r})^2 + 2\sqrt{c} + c > (\sqrt{S} - \sqrt{s})^2 + 2\sqrt{c} + c \\ &= S + s + Ss = \rho_B - 1, \end{aligned}$$

and Corollary 2 b) yields the assertion. $\square$

Next, we extend Theorem 1 b) to the case of n complex disks. The proof proceeds similar to that of Theorem 1 in [4].

Theorem 6. *Let $n \in \mathbb{N}$ and $r, s \in \mathbb{R}^n_{\ge 0}$ be given such that $s_1 \ge \cdots \ge s_n$. For $k \in \{1, \ldots, n\} =: [n]$ define $A_k := \prod_{j=1}^k D(1, r_j)$ and $B_k := \prod_{j=1}^k D(1, s_j)$, and let $\rho_{A_k} := \prod_{j=1}^k (1 + r_j)$ and $\rho_{B_k} := \prod_{j=1}^k (1 + s_j)$ denote the rightmost real points of A_k and B_k, respectively. Then, the following equivalence relations holds true:*

$$\forall k \in [n] : B_k \subseteq A_k \quad \Leftrightarrow \quad \forall k \in [n] : \rho_{B_k} \in A_k \quad \Leftrightarrow \quad \forall k \in [n] : \rho_{B_k} \le \rho_{A_k}$$

Proof. Again, the implications

$$\forall k \in [n] : B_k \subseteq A_k \quad \Rightarrow \quad \forall k \in [n] : \rho_{B_k} \in A_k \quad \Rightarrow \quad \forall k \in [n] : \rho_{B_k} \le \rho_{A_k}$$

are trivial and we are left with proving

$$\forall k \in [n] : \rho_{B_k} \le \rho_{A_k} \quad \Rightarrow \quad \forall k \in [n] : B_k \subseteq A_k.$$

Therefore, let us assume that the premise holds true:

$$\forall k \in [n] : \quad \prod_{j=1}^k (1 + s_j) \le \prod_{j=1}^k (1 + r_j) \tag{13}$$

The proof proceeds by induction on n. For $n = 1$ we have $B_1 = D(1, s_1) \subseteq D(1, r_1) = A_1$ because $s_1 \le r_1$ by (13). Now, let $n > 1$ and suppose that the assertion holds true for $n - 1$. Thus, it remains to prove that $B := B_n \subseteq A_n =: A$. If $s_j \le r_j$ for all $j = 1, \ldots, n$, then the assertion is trivial. Thus, we may assume that there is a smallest index $m \in \{1, \ldots, n\}$ such that $s_m > r_m$.

Since $s_1 \le r_1$ by (13) for $k = 1$, we have $m \ge 2$. The minimal choice of m and the assumption $s_{m-1} \ge s_m$ yield

$$r_{m-1} \quad \ge \quad s_{m-1} \ge s_m > r_m. \tag{14}$$

For $j \in \{1, \ldots, n-1\}$ define

$$s'_j \quad := \quad \begin{cases} s_j & \text{if } \ 1 \le j \le m-1 \\ s_{j+1} & \text{if } \ m \le j \le n-1 \end{cases}$$

$$r'_j \quad := \quad \begin{cases} r_j & \text{if } \ 1 \le j \le m-2 \\ \dfrac{(1+r_{m-1})(1+r_m)}{1+s_m} - 1 & \text{if } \ j = m-1 \\ r_{j+1} & \text{if } \ m \le j \le n-1. \end{cases}$$

Clearly, $s'_1 \ge \cdots \ge s'_{n-1}$ and $\prod_{j=1}^{k}(1+s'_j) \le \prod_{j=1}^{k}(1+r'_j)$ for $k \in \{1, \ldots, m-2\}$. For $k \in \{m-1, \ldots, n-1\}$ assumption (13) implies

$$\begin{aligned} \prod_{j=1}^{k}(1+s'_j) \quad &= \quad \frac{1}{1+s_m}\prod_{j=1}^{k+1}(1+s_j) \\ &\le \quad \frac{(1+r_{m-1})(1+r_m)}{1+s_m}\prod_{j=1}^{m-2}(1+r_j)\prod_{j=m+1}^{k+1}(1+r_j) \\ &= \quad \prod_{j=1}^{k}(1+r'_j). \end{aligned}$$

The induction hypothesis applied to $r', s' \in \mathbb{R}^{n-1}$ supplies

$$B' := \prod_{j=1}^{n-1} D(1, s'_j) \subseteq \prod_{j=1}^{n-1} D(1, r'_j) =: A'.$$

From (14) it follows that

$$s_m \in \ [r_m, r_{m-1}] \quad \text{and} \quad r'_{m-1} = \frac{(1+r_{m-1})(1+r_m)}{1+s_m} - 1 \ \in \ [r_m, r_{m-1}].$$

By Theorem 1 b)

$$D(1, s_m)D(1, r'_{m-1}) \subseteq D(1, r_{m-1})D(1, r_m)$$

holds true and we conclude

$$\begin{aligned} B \quad &= \quad D(1, s_m)B' \subseteq D(1, s_m)A' = D(1, s_m)D(1, r'_{m-1}) \prod_{\substack{j=1 \\ j \ne m-1}}^{n-1} D(1, r'_j) \\ &\subseteq \quad D(1, r_{m-1})D(1, r_m) \prod_{\substack{j=1 \\ j \ne m-1, m}}^{n} D(1, r_j) = A. \end{aligned}$$

$\square$

Condition (13) is sufficient for deciding $B \subseteq A$ but by Theorem 6 it already implies that all partial disk products B_k are contained in the corresponding A_k, $k = 1, \dots, n$. It is not clear how to formulate a criterion for $B = B_n$ being a subset of $A = A_n$ such that $B_k \subseteq A_k$ must not necessarily hold true for all $k < n$.

Finally we remark that, analogously to Corollary 2, it is easy to show that $B \subseteq \overset{\circ}{A}$ holds true if at least one of the n inequalities in (13) is strict.

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