THE DETERMINANT OF A COMPLEX MATRIX AND GERSHGORIN CIRCLES*

FLORIAN BÜNGER † AND SIEGFRIED M. RUMP ‡

Abstract. Each connected component of the Gershgorin circles of a matrix contains exactly as many eigenvalues as circles are involved. Thus the Minkowski (set) product of all circles contains the determinant if all circles are disjoint. In [3] we proved that statement to be true for real matrices whose circles need not to be disjoint. Moreover we asked whether the statement remains true for complex matrices. This note answers that in the affirmative. As a by-product we derive a parameterization of the outer loop of a Cartesian oval without case distinction.

Key words. Gershgorin circle, determinant, Minkowski product.

AMS subject classifications. 15A15, 15A42

The Gershgorin region G of an $n \times n$ complex matrix A, i.e., the union of all its Gershgorin disks, contains all its eigenvalues, so there are always n points in G whose product is $\det(A)$. In the case of overlapping Gershgorin circles, one may ask whether these points can be chosen so that one is in each Gershgorin circle of A. In [3] this was answered in the affirmative for real A, where in fact all points can be chosen to be real.

In this note we prove that also for a complex matrix the set product of the Gershgorin circles contains the determinant. The proof follows, using a result by Hans Schneider, as a corollary of the following theorem.

THEOREM 1. *Let $r \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}^n$ with $|\lambda_1| \geq \cdots \geq |\lambda_n|$ be given. Suppose*

$$\prod_{j=1}^{k} |\lambda_j| \leq \prod_{j=1}^{k} r_j \quad \text{for} \quad 1 \leq k \leq n. \tag{1}$$

Then there exists $g \in \mathbb{C}^n$ with $|g_j| \leq r_j$ for $1 \leq j \leq n$ and

$$\prod_{j=1}^{n} (1+\lambda_j) = \prod_{j=1}^{n} (1+g_j) \ . \tag{2}$$

For the proof we need some preparations. First, we characterize the shape of the Minkowski (set) product of two complex circles by a new parameterization without case distinction on the size of the radii. Next we need a lemma how the radii of the circles can be changed such that their product becomes a superset of the previous. Based on that Theorem 1 will be proved, and the announced result on Gershgorin circles follows as a corollary.

The Minkowski product of two complex circles is well known to be bounded by a Cartesian oval [1, Proposition 5]. For two complex circles with radii R and r and common center 1, this oval can be characterized [1, (17)] as the set of points $x + iy$ in the Gaussian plane with

$$[(x-1)^2 + y^2 + R^2 - r^2 - R^2 r^2]^2 = 4R^2[(x-1+r^2)^2 + y^2]. \tag{3}$$

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†Institute for Reliable Computing, Hamburg University of Technology, Am Schwarzenberg-Campus 3, Hamburg 21071, Germany (florian.buenger@tuhh.de).

‡Institute for Reliable Computing, Hamburg University of Technology, Am Schwarzenberg-Campus 3, Hamburg 21071, Germany, and Visiting Professor at Waseda University, Faculty of Science and Engineering, 3–4–1 Okubo, Shinjuku-ku, Tokyo 169–8555, Japan (rump@tuhh.de).

Cartesian ovals may have different shapes, some of which are shown in Figure 1. In this note we are interested

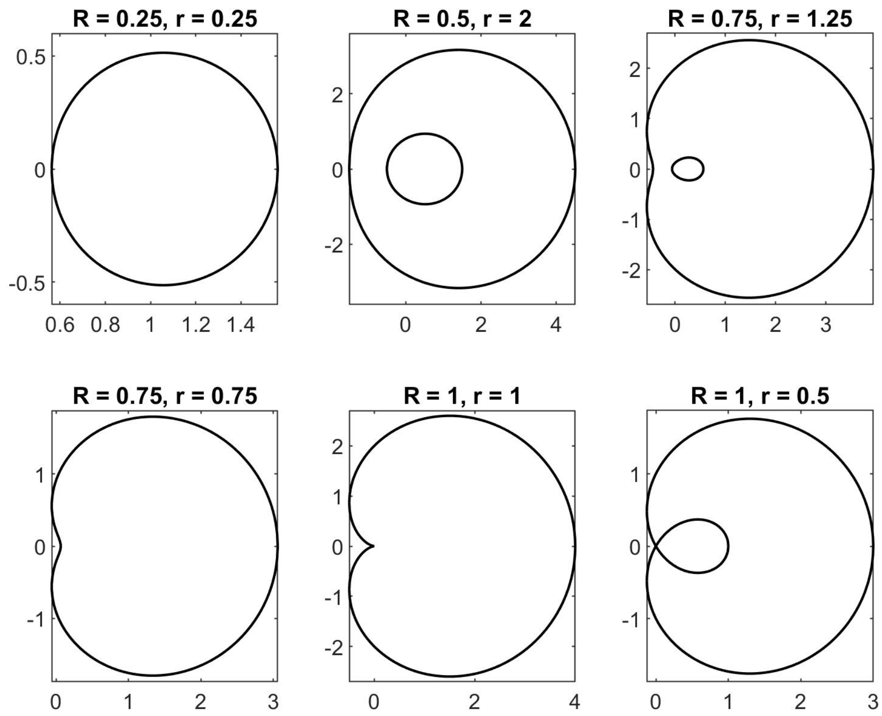

FIGURE 1. *Cartesian ovals for products of complex circles with center 1 and radii R and r.*

in the set product of the discs, the boundary of which is the outer loop of a Cartesian oval. To our knowledge, known parameterizations of the outer loop contain some case distinctions on the radii R and r. Following is a parameterization without case distinctions.

LEMMA 2. *Let $R \geq r \in \mathbb{R}_{\geq 0}$ be given. For $\zeta \in \mathbb{C}$ and $\rho \in \mathbb{R}_{\geq 0}$ define*

$$D(\zeta, \rho) := \{z \in \mathbb{C} : |z - \zeta| \leq \rho\}.$$

Then the boundary of the Minkowski (set) product $D(1, R) \cdot D(1, r)$ is parameterized by

$$x(t) := \frac{t^2 + 1 - (R^2 + r^2)}{2} \tag{4}$$

$$y(t) := \pm\sqrt{(R + tr)^2 - (x(t) - 1 + r^2)^2} \tag{5}$$

$$t \in [t_1, t_2] := [\ |1 - R| - r\ ,\ 1 + R + r\ ]. \tag{6}$$

Proof. We assume $R > r$; the case $R = r$ follows by continuity. We use a bipolar representation for the Cartesian oval (3). By [2], p. 48, its outer loop consists of all complex points $z = x + iy$ for which the distances $\rho_1 := |z - a_1|$ and $\rho_2 := |z - a_2|$ to the two poles $a_1 := 1 - r^2$ and $a_2 := 1 - R^2$ fulfill

$$R\rho_1 - r\rho_2 = a_1 - a_2 = R^2 - r^2. \tag{7}$$

The triangle inequality imposes the following constraints for ρ_1 and ρ_2, cf. [2, (2.4)]:

$$\rho_1 + \rho_2 = |z - a_1| + |z - a_2| \geq |a_1 - z + z - a_2| = a_1 - a_2 = R^2 - r^2 \tag{8}$$

$$|\rho_1 - \rho_2| = ||z - a_1| - |z - a_2|| \leq |a_1 - z - (z - a_2)| = a_1 - a_2 = R^2 - r^2. \tag{9}$$

Clearly, the linear equation (7) for the vector $(\rho_1, \rho_2) \in \mathbb{R}^2$ implies

$$(\rho_1, \rho_2) = (R, r) + t(r, R) = (R + tr, r + tR) \tag{10}$$

for some $t \in \mathbb{R}$. From (8), (9), (10), and $R - r > 0$ it follows that

$$\begin{aligned} R^2 - r^2 \leq \rho_1 + \rho_2 = (1 + t)(r + R) \quad &\Leftrightarrow \quad R - 1 - r \leq t \\ R^2 - r^2 \geq |\rho_1 - \rho_2| = |1 - t|(R - r) \quad &\Leftrightarrow \quad |1 - t| \leq R + r \\ &\Leftrightarrow \quad 1 - R - r \leq t \leq 1 + R + r. \end{aligned}$$

Combining both gives the parameter interval

$$t \in [\ \max\{R - 1 - r, 1 - R - r\}, 1 + R + r\ ] = [\ |1 - R| - r, 1 + R + r\ ]. \tag{11}$$

This is (6). Next,

$$\begin{aligned} (t^2 - 1)(R^2 - r^2) &= (r + tR)^2 - (R + tr)^2 = \rho_2^2 - \rho_1^2 = |z - a_2|^2 - |z - a_1|^2 \\ &= (x - a_2)^2 - (x - a_1)^2 = (a_1 - a_2)(2x - (a_1 + a_2)) \\ &= 2(R^2 - r^2)\left(x - 1 + \frac{R^2 + r^2}{2}\right) \end{aligned}$$

yields

$$x = \frac{t^2 + 1}{2} - \frac{R^2 + r^2}{2}\ .$$

This is (4). Finally, $y^2 = \rho_1^2 - (x - a_1)^2 = (R + tr)^2 - (x - 1 + r^2)^2$ proves (5). □

The extremal real points of $M := D(1, R) \cdot D(1, r)$ are well-known to be

$$\begin{aligned} a &:= \min M \cap \mathbb{R} = \min\,(1 - R)(1 \pm r) \\ b &:= \max M \cap \mathbb{R} = (1 + R)(1 + r). \end{aligned}$$

This is easily seen by setting $y = 0$ in (3). It corresponds to the parameterization (4)-(6) by inserting the extremal values for t, in other words the curve endpoints a and b are connected when t varies from t_1 to t_2.

LEMMA 3. *Let $R \geq S \geq s \geq r \in \mathbb{R}_{\geq 0}$ be given such that $Rr = Ss$. Then*

$$D(1, S) \cdot D(1, s) \subseteq D(1, R) \cdot D(1, r) \tag{12}$$

using Minkowski (set) products.

Proof. Set $M := D(1, R) \cdot D(1, r)$, $\tilde{M} := D(1, S) \cdot D(1, s)$, and $c := Rr = Ss$. We exclude the trivial cases by henceforth assuming $(R, r) \neq (S, s)$ and $c \neq 0$.

By Lemma 2 the outer loop of M has the parameterization (4)-(6), and analogously, $\tilde{x}, \tilde{y}, \tilde{t}_1, \tilde{t}_2$ are defined for $\tilde{M}$ by replacing R, r by S, s. Now, a computation gives

$$x(t)^2 + y(t)^2 = (t+c)^2 \quad \text{for all } t \in [t_1, t_2], \tag{13}$$

$$\tilde{x}(t)^2 + \tilde{y}(t)^2 = (t+c)^2 \quad \text{for all } t \in [\tilde{t}_1, \tilde{t}_2]. \tag{14}$$

By $(R, r) \neq (S, s)$ we have $S - s < R - r$ and $\sqrt{S} - \sqrt{s} < \sqrt{R} - \sqrt{r}$. This implies

$$S^2 + s^2 = (S-s)^2 + 2c \;<\; (R-r)^2 + 2c = R^2 + r^2 \tag{15}$$

$$S + s = (\sqrt{S} - \sqrt{s})^2 + 2\sqrt{c} \;<\; (\sqrt{R} - \sqrt{r})^2 + 2\sqrt{c} = R + r. \tag{16}$$

The sets M and $\tilde{M}$ have the common point 1, and $(1+R)(1+r)$ is in M but not in $\tilde{M}$ because $\tilde{z} \in \tilde{M} = D(1, S) \cdot D(1, s)$, $Ss = Rr$, and (16) imply

$$|\tilde{z}| \leq (1+S)(1+s) < (1+R)(1+r).$$

Hence (12) follows if the boundary curves of M and $\tilde{M}$ have no intersection. In order to derive a contradiction we assume that there exists an intersection point $x(\tau) + iy(\tau) = \tilde{x}(\tilde{\tau}) + i\tilde{y}(\tilde{\tau})$ for some $\tau \in [t_1, t_2]$ and $\tilde{\tau} \in [\tilde{t}_1, \tilde{t}_2]$. From (13) and (14) it follows that $(\tau + c)^2 = (\tilde{\tau} + c)^2$, i.e., $\tilde{\tau} = \tau$ or $\tilde{\tau} = -\tau - 2c$. By $x(\tau) = x(\tilde{\tau})$, (4) and (15),

$$\tilde{\tau}^2 - \tau^2 = S^2 + s^2 - (R^2 + r^2) < 0. \tag{17}$$

This excludes $\tilde{\tau} = \tau$ wherefore $\tilde{\tau} = -(\tau + 2c)$, such that (17) becomes

$$\tau + c = \frac{S^2 + s^2 - (R^2 + r^2)}{4c} < 0.$$

Thus, $0 > \tau + c \geq t_1 + c = |1 - R| - r + c$ yields $r(1-R) = r - c > |1 - R| \geq 0$. Hence $1 > R > r$ which gives the contradiction $r(1-R) > |1 - R| = 1 - R > r(1-R)$. $\square$

Proof of Theorem 1. For $n = 1$ choose $g_1 := \lambda_1$. We proceed by induction and suppose that the assertion is true for $n-1$. If $|\lambda_j| \leq r_j$ for all $j = 1, \ldots, n$, then we can choose $g_j := \lambda_j$ to prove the assertion. Thus, we may assume that there is a smallest index $m \in \{1, \ldots, n\}$ such that $|\lambda_m| > r_m$. Assumption (1) for $k = 1$ implies $m \geq 2$. The minimal choice of m and the assumption $|\lambda_{m-1}| \geq |\lambda_m|$ yield

$$r_{m-1} \geq |\lambda_{m-1}| \geq |\lambda_m| > r_m. \tag{18}$$

For $j \in \{1, \ldots, n-1\}$ define

$$\lambda'_j := \begin{cases} \lambda_j & \text{if } 1 \leq j \leq m-1 \\ \lambda_{j+1} & \text{if } m \leq j \leq n-1 \end{cases} \qquad \text{and} \qquad r'_j := \begin{cases} r_j & \text{if } 1 \leq j \leq m-2 \\ \dfrac{r_{m-1} r_m}{|\lambda_m|} & \text{if } j = m-1 \\ r_{j+1} & \text{if } m \leq j \leq n-1. \end{cases}$$

Clearly, $|\lambda'_1| \geq \cdots \geq |\lambda'_{n-1}|$ and $\prod_{j=1}^k |\lambda'_j| \leq \prod_{j=1}^k r'_j$ for $k \in \{1, \ldots, m-2\}$. For $k \in \{m-1, \ldots, n-1\}$ assumption (1) implies

$$\prod_{j=1}^{k} |\lambda'_j| = \frac{1}{|\lambda_m|} \prod_{j=1}^{k+1} |\lambda_j| \leq \frac{r_{m-1} r_m}{|\lambda_m|} \prod_{j=1}^{m-2} r_j \prod_{j=m+1}^{k+1} r_j = \prod_{j=1}^{k} r'_j.$$

The induction hypothesis applied to $r' \in \mathbb{R}^{n-1}$ and $\lambda' \in \mathbb{C}^{n-1}$ supplies $g' \in \mathbb{C}^{n-1}$ such that

$$\prod_{\substack{j=1\\ j\neq m}}^{n}(1+\lambda_j) = \prod_{j=1}^{n-1}(1+\lambda'_j) = \prod_{j=1}^{n-1}(1+g'_j) \quad \text{and} \quad |g'_j| \leq r'_j \ , \ j=1,\ldots,n-1. \tag{19}$$

From (18) and $r'_{m-1} = \dfrac{r_{m-1}r_m}{|\lambda_m|}$ it follows that

$$|\lambda_m|, r'_{m-1} \in \ [r_m, r_{m-1}].$$

By Lemma 3,

$$D(1,|\lambda_m|)D(1,r'_{m-1}) \subseteq D(1,r_{m-1})D(1,r_m)$$

so that $(1+\lambda_m)(1+g'_{m-1}) = (1+g_{m-1})(1+g_m)$ for suitable $g_{m-1}, g_m \in \mathbb{C}$ with $|g_{m-1}| \leq r_{m-1}$ and $|g_m| \leq r_m$. For $j \in \{1,\ldots,n\}\backslash\{m-1,m\}$ define

$$g_j := \begin{cases} g'_j & \text{if } \ 1 \leq j \leq m-2 \\ g'_{j-1} & \text{if } \ m+1 \leq j \leq n. \end{cases}$$

Then, using (19), we obtain

$$\prod_{j=1}^{n}(1+\lambda_j) = (1+\lambda_m)(1+g'_{m-1}) \prod_{\substack{j=1\\ j\neq m-1}}^{n-1}(1+g'_j) = \prod_{j=1}^{n}(1+g_j)$$

and $|g_j| \leq r_j$ for all $j \in \{1,\ldots,n\}$. ∎

Corollary 4. *For a complex $n \times n$ matrix A, there exist g_j in the j-th Gershgorin circle for $1 \leq j \leq n$ with*

$$\det(A) = \prod_{j=1}^{n} g_j. \tag{20}$$

Proof. We proceed as in the proof of Theorem 1 in [3], the pendant of our result for real matrices, and note that if some diagonal element of A is zero, then the set product of the Gershgorin discs is a disc centered at the origin with radius equal to the product of the ℓ_1-norms of the rows of A. In this case Hadamard's bound

$$|\det(A)| \leq \prod_{j=1}^{n}\|A_{j*}\|_2 \leq \prod_{j=1}^{n}\|A_{j*}\|_1 \tag{21}$$

proves the result. Henceforth we assume without loss of generality that $a_{ii} \neq 0$ for $1 \leq i \leq n$. Denote the row sum of absolute values of the off-diagonal elements of a matrix A by $R_j(A) := \sum_{k\neq j}|a_{jk}|$, so that the j-th Gershgorin circle of A is $G_j(A) = \{z : |a_{jj} - z| \leq R_j(A)\}$.

Denote the diagonal of A by D, so that $D^{-1}A = I + E$ splits into the identity matrix I and the matrix E with zero diagonal elements. With suitable ordering of the eigenvalues λ_j of E and the $R_j(E)$, we may assume without loss of generality

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \qquad \text{and} \qquad R_1(E) \geq R_2(E) \geq \cdots \geq R_n(E).$$

Hans Schneider [4, Theorem 1] proved

$$\prod_{j=1}^{k} |\lambda_j| \le \prod_{j=1}^{k} R_j(E) \qquad \text{for } 1 \le k \le n, \tag{22}$$

so that Theorem 2 implies existence of $|g_j| \le R_j(E)$ with

$$\det(I+E) = \prod_{j=1}^{n} (1+\lambda_j) = \prod_{j=1}^{n} (1+g_j). \tag{23}$$

Hence

$$\begin{aligned}
\det(A) &= \prod_{j=1}^{n} a_{jj}(1+\lambda_j) \in \prod_{j=1}^{n} \{a_{jj}z : |1-z| \le R_j(E)\} \\
&= \prod_{j=1}^{n} \{z : |1-a_{jj}^{-1}z| \le R_j(E)\} = \prod_{j=1}^{n} \{z : |a_{jj}-z| \le |a_{jj}|R_j(E)\} \\
&= \prod_{j=1}^{n} \{z : |a_{jj}-z| \le R_j(A)\} = \prod_{j=1}^{n} G_j(A)
\end{aligned}$$

finishes the proof. ∎

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