Computational Experience with Rigorous Error Bounds for the Netlib Linear Programming Library

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Abstract. The Netlib library of linear programming problems is a well known suite containing many real world applications. Recently it was shown by Ordóñez and Freund that 71% of these problems are ill-conditioned. Hence, numerical difficulties may occur. Here, we present rigorous results for this library that are computed by a verification method using interval arithmetic. In addition to the original input data of these problems we also consider interval input data. The computed rigorous bounds and the performance of the algorithms are related to the distance to the next ill-posed linear programming problem.

Keywords: linear programming, interval arithmetic, rigorous error bounds, sensitivity analysis, Netlib linear programming library

AMS Subject classification: 90C05, 65G30, 65N15

1. Introduction

The Netlib suite of linear optimization problems [15] includes many real world applications like stochastic forestry problems, oil refinery problems, flap settings of aircraft, pilot models, audit staff scheduling, truss structure problems, airline schedule planning, industrial production and allocation models, image restoration problems, and multisector economic planning problems. It contains problems ranging in size from 32 variables and 27 constraints up to 15695 variables and 16675 constraints.

In a recent publication Ordóñez and Freund have shown 71% of the Netlib problems to be ill-conditioned [20]. Only 19% of the problem instances remain ill-conditioned after preprocessing techniques are applied. In a paper by Fourer and Gay [5], however, it is observed that rounding errors in presolve may change the status of a linear program from feasible to infeasible, and vice versa.

The goal of this paper is to present verified numerical results for both the unprocessed original input data of this library and interval input data. These results are obtained by a verification method for linear programming problems that is presented in [9]. Our central observation is that, roughly spoken, rigorous error bounds together with certificates of feasibility are obtained for the well-posed problem instances. Further-

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more, for many ill-posed problem instances a rigorous lower or upper bound of the optimal value can be computed. The computational costs increase with decreasing distances to primal or dual infeasibility, that is with decreasing distance to ill-posedness.

We mention that the algorithms used to compute these rigorous bounds are not only useful for linear programming problems. They can also be used in global optimization and mixed-integer programming whenever linear relaxations must be solved (see for example Floudas [4] and Neumaier [18]). Thus safe results can be obtained for these nonlinear problems.

This paper is organized as follows. Section 2 contains the basic theorems of our verification method. In Section 3 a brief introduction to condition numbers for linear programming problems is given. The numerical results of our computations are discussed in Section 4. Finally, in Section 5, some concluding remarks are given. The tables with the numerical results are presented in the appendix.

2. Rigorous Error Bounds

We consider the linear programming problem

$$\begin{aligned} f^* &:= \min_{x \in X} c^T x \\ X &:= \{x \in \mathbb{R}^n : \; Ax \leq a, \; Bx = b, \; \underline{x} \leq x \leq \overline{x}\}, \end{aligned} \tag{1}$$

with f^* becoming $+\infty$ if X is empty.

The input parameter are

$$P = (A, B, a, b, c) \in \mathbb{R}^{(m+p+1)n+m+p},$$

where A is a real $m \times n$ matrix, B a real $p \times n$ matrix, $c, x \in \mathbb{R}^n$, $a \in \mathbb{R}^m$, and $b \in \mathbb{R}^p$. Further, the simple bounds $\underline{x} \leq \overline{x}$, which may be infinite; that is $\underline{x}_j := -\infty$ or $\overline{x}_j := +\infty$ for some $j \in \{1, \ldots, n\}$. The set of indices where the simple bounds are both infinite is denoted by

$$J^\infty := \{j \in \{1, \ldots, n\} : \; \underline{x}_j = -\infty, \text{ and } \overline{x}_j = +\infty\}$$

and its complement by $J^r := \{1, \ldots, n\} \setminus J^\infty$.

The input data may be uncertain. We describe these uncertainties by considering a family of linear programming problems P, where $P \in \mathbf{P}$ and $\mathbf{P}$ represents the corresponding interval quantities. We require only some elementary facts about interval arithmetic, which can be found for example in Alefeld and Herzberger [1], Hansen and Walster [7], Kearfott [10], Moore [14], and Neumaier [16], [17].

To compute a rigorous upper bound, the basic idea is to determine an interval vector $\mathbf{x}$ that contains a feasible solution for every $P \in \mathbf{P}$, being in the relative interior of X. This solution should be close to an optimal solution but sufficiently far away from degeneracy and infeasibility. The next theorem gives favourable characteristics of $\mathbf{x}$.

THEOREM 1. *Let* $\mathbf{P} := (\mathbf{A}, \mathbf{B}, \mathbf{a}, \mathbf{b}, \mathbf{c})$ *be a family of lp-problems with input data* $P \in \mathbf{P}$ *and simple bounds* $\underline{x} \leq \overline{x}$*. Suppose that there exists an interval vector* $\mathbf{x} \in \mathbb{IR}^n$ *such that*

$$\mathbf{A}\mathbf{x} \leq \mathbf{a}, \quad \underline{x} \leq \mathbf{x} \leq \overline{x},$$

and

$$\forall\, B \in \mathbf{B},\ b \in \mathbf{b}\ \exists\, x \in \mathbf{x}:\ Bx = b.$$

Then for every $P \in \mathbf{P}$ *there exists a primal feasible solution* $x(P) \in \mathbf{x}$*, and the inequality*

$$\sup_{P \in \mathbf{P}} f^*(P) \quad \leq \quad \overline{f^*} := \max\{\mathbf{c}^T \mathbf{x}\} \tag{2}$$

is satisfied. Moreover, if the objective function is bounded from below for every lp-problem with input data $P \in \mathbf{P}$*, then each problem has an optimal solution.*

The following theorem provides the basic characteristics of a rigorous lower bound.

THEOREM 2. *Let* $\mathbf{P} := (\mathbf{A}, \mathbf{B}, \mathbf{a}, \mathbf{b}, \mathbf{c})$ *be a family of lp-problems with input data* $P \in \mathbf{P}$ *and simple bounds* $\underline{x} \leq \overline{x}$*. Suppose that there exist interval vectors* $\mathbf{y} \in \mathbb{IR}^m$ *and* $\mathbf{z} \in \mathbb{IR}^p$ *such that*

(i) the sign condition

$$\mathbf{y} \leq 0$$

holds true,

(ii) for $j \in J^\infty$ *the equations*

$$\forall\, A \in \mathbf{A}, B \in \mathbf{B}, c \in \mathbf{c}\ \exists\, y \in \mathbf{y}, z \in \mathbf{z}:$$
$$(A_{:j})^T y + (B_{:j})^T z = c_j$$

are fulfilled,

(iii) and for $j \in J^r$ *the intervals*

$$\mathbf{d}_j := \mathbf{c}_j - (\mathbf{A}_{:j})^T \mathbf{y} - (\mathbf{B}_{:j})^T \mathbf{z} \tag{3}$$

satisfy the inequalities

$$\mathbf{d}_j \leq 0,\ \ \text{if } \underline{x}_j = -\infty$$
$$\mathbf{d}_j \geq 0,\ \ \text{if } \overline{x}_j = +\infty.$$

Then the inequality

$$\inf_{P \in \mathbf{P}} f^*(P) \quad \geq \quad \underline{f^*} := \min\{\mathbf{a}^T\mathbf{y} + \mathbf{b}^T\mathbf{z} + \sum_{\substack{j \in J^r \\ \overline{\mathbf{d}}_\mathbf{j} > 0}} \underline{x}_j \mathbf{d}_j^+ + \sum_{\substack{j \in J^r \\ \underline{\mathbf{d}}_\mathbf{j} < 0}} \overline{x}_j \mathbf{d}_j^-\} \quad (4)$$

is fulfilled, and $\underline{f^*}$ *is a finite lower bound of the global minimum value. Moreover, if*

(a) all input data are point data (i.e. $P = \mathbf{P}$*),*

(b) P *has an optimal solution* (y^*, z^*, u^*, v^*)*,*

(c) $\mathbf{y} := y^*$*,* $\mathbf{z} := z^*$*,*

(d) the quantities in (3) *and* (4) *are calculated exactly,*

then the conditions (i),(ii) and (iii) are satisfied, and the optimal value $f^*(P) = \underline{f^*}$*; that is, this lower error bound is sharp for point input data and exact computations.*

For proofs of these two theorems as well as algorithms for computing appropriate interval vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ the reader is referred to [9]. Roughly spoken, the algorithms try to find these interval vectors in an iterative manner by computing approximate solutions of perturbed linear programming problems. In the special case where all simple bounds are finite the conditions (ii) and (iii) of Theorem 2 are trivially satisfied. Hence, for each nonnegative interval vector $\mathbf{y}$ the right hand side of (4) delivers a rigorous lower bound in $O(n^2)$ operations.

We emphasize that the previous analysis gives a rigorous certificate for the existence of optimal solutions if both bounds $\underline{f^*}$ and $\overline{f^*}$ are finite. Our numerical experience is that these bounds can be computed also for degenerate and ill-conditioned problems provided the linear programming solver has computed sufficiently accurate results. The quality of $\underline{f^*}$ and $\overline{f^*}$ depends mainly on the accuracy of the computed approximations.

A generalization of our approach to convex problems is described in [8].

In [11] the algorithms presented in [9] were implemented using the interval library PROFIL/BIAS [12]. There the reader can find primary numerical results for the Netlib suite. The linear programming solver used to compute approximate solutions was lp_solve [3].

Recently, Neumaier and Shcherbina [19] have investigated rigorous error bounds for mixed-integer linear programming problems. In their paper, in addition to rigorous cuts and a certificate of infeasibility, a rigorous lower bound for linear programming problems with exact input data and finite simple bounds is presented. Our focus is on problems with uncertain input data and simple bounds which may be infinite. In the overlapping part of both papers, where the simple bounds are finite, both rigorous lower bounds coincide.

Beeck [2], Krawzcyk [13], and Rump [21] have developed methods for computing rigorous error bounds for lp-problems where the optimal solution is unique. They use the simplex method for the computation of an optimal basic index set. Then with interval methods the optimality of this index set is verified a posteriori, and rigorous error bounds for the optimal vertex and the optimal value are calculated. These methods are more expensive, they require $O(n^3)$ operations even for finite simple bounds, and they can only be applied to non-degenerate problems. Furthermore, the distances to primal and dual infeasibility must be greater than 0. Since most problems of the Netlib lp library do not have these properties, the mentioned methods do not work.

3. Condition Numbers

In order to define the condition number for linear programming problems, Ordóñez and Freund used the *ground-set* format

$$\begin{aligned} f^*(P) &:= \min_{x \in X(P)} c^T x \\ X(P) &:= \{x \in \mathbb{R}^n : \; Ax - b \in C_Y, \; x \in S\}. \end{aligned} \tag{5}$$

A specific ground-set problem is defined by its input data $P = (A, b, c)$ consisting of the real $m \times n$ matrix A and the real vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. The set $C_Y \subseteq \mathbb{R}^m$ is a closed convex cone, and the set $S \subseteq \mathbb{R}^n$ is closed and convex.

The corresponding *dual problem* is

$$\begin{aligned} f^*(P) &:= \max_{(y,v) \in Y(P)} b^T y - v \\ Y(P) &:= \{(y, v) \in \mathbb{R}^{m \times n} : \; (c - A^T y, v) \in C_S^*, \; y \in C_Y^*\}. \end{aligned} \tag{6}$$

Here, C_Y^* denotes the *dual cone* of C_Y, i.e.

$$C_Y^* := \{y \in \mathbb{R}^m : \; z^T y \geq 0 \text{ for all } z \in C_Y\},$$

and C_S^* denotes the dual cone of

$$C_S := \{(x, t) : \; x \in tS \text{ and } t > 0\}.$$

The distances to primal and dual infeasibility are defined by

$$\rho_p(P) := \inf\{\|\Delta P\| : \; X(P + \Delta P) = \emptyset\} \tag{7}$$

$$\rho_d(P) := \inf\{\|\Delta P\| : \; Y(P + \Delta P) = \emptyset\}, \tag{8}$$

where $\Delta P := (\Delta A, \Delta b, \Delta c)$ and

$$\|\Delta P\| := \max\{\|\Delta A\|, \|\Delta b\|_1, \|\Delta c\|_\infty\},$$

with $\|\Delta A\|$ denoting the corresponding operator norm.

Using this choice of norms, Ordóñez and Freund have shown that the distances to infeasibility can be computed by solving $2n+2m$ linear programming problems of size roughly that of the original problem.

The condition number of a linear programming problem is defined as the quotient of the norm of the input data and the smallest distance to infeasibility,

$$\mathrm{cond}(P) := \frac{\|P\|}{\min\{\rho_p(P), \rho_d(P)\}}. \tag{9}$$

A problem is called ill-posed if $\min\{\rho_p(P), \rho_d(P)\} = 0$ or equivalently $\mathrm{cond}(P) = \infty$.

We mention that Ordóñez and Freund have computed the condition numbers for the problems in the Netlib lp library, but not rigorously.

Our linear programming format (1) can be described in the ground-set format by aggregating the equality and inequality constraints to

$$\begin{pmatrix} A \\ B \end{pmatrix} x - \begin{pmatrix} a \\ b \end{pmatrix} \in C_Y := \begin{pmatrix} \mathbb{R}^m_- \\ 0 \end{pmatrix}$$

and using $S := \{x \in \mathbb{R}^n : \underline{x} \le x \le \overline{x}\}$. This transformation yields the condition number for our format.

4. Numerical Results

In the following we present our rigorous results for the Netlib suite of linear programming problems. The implementation is a slight modification of the algorithms in [9] with respect to the deflation parameter vector ε. The linear programming solver used to compute approximate solutions was lp_solve 5.5 [3]. Compiler suite was the gcc version 3.3.1 [6]. All computations were done on a PC with 2.8 GHz.

In order to compare the results, we have chosen exactly the set of problems that Ordóñez and Freund [20] have computed condition numbers for. For the problems *DEGEN3* and *PILOT*, lp_solve was aborted because the original problem had not been solved after 24 hours, leaving 87 problems in the test set.

Our results are displayed in the appendix. Summarizing, it can be seen that in almost all cases rigorous upper bounds $\overline{f^*}$ and rigorous lower bounds $\underline{f^*}$ are computed if the distance to respectively primal infeasibility ρ_p and dual infeasibility ρ_d is greater than 0. Rigorous bounds and a certificate of the existence of optimal solutions are obtained for well-posed problems.

Table I shows the accuracy of the rigorous bounds. The first column contains the name of the problem. Columns two and three contain the distances to infeasibility ρ_d and ρ_p as computed by Ordóñez and Freund. Then the lower and upper bound rounded to 5 decimal digits of precision are displayed, and the last column contains the relative error

$$\mu := \frac{|\overline{f^*} - \underline{f^*}|}{\max\{1,\, 0.5(|\overline{f^*}| + |\underline{f^*}|)\}}.$$

If one of the bounds in the quotient μ could not be computed, it is substituted by the approximate optimal value delivered by lp_solve. In the case of both bounds being infinite, μ is set to NaN (i.e., Not a Number).

Throughout our experiments we have used lp_solve 5.5 with only the default optimization parameters. From Table I it can be seen that for almost all problems the relative error μ varies between $1 \cdot 10^{-8}$ and $1 \cdot 10^{-16}$. With lp_solve's default stopping tolerance of $1 \cdot 10^{-9}$, this is about the best one could expect.

We see that in almost all cases the rigorous lower and upper bound is finite if the distance to dual and primal infeasibility is greater than 0, respectively. Only the problems *SCSD8* and *SCTAP1* deliver no upper bound despite a primal distance to infeasibility greater than 0. On the other hand, the problems *25FV47, 80BAU3B, BEACONFD, BNL2, CYCLE, D2Q06C, E226, RECIPE, SCRS8, STANDGUB* deliver a lower bound and *80BAU3B, ADLITTLE, E226, FINNIS, GFRD-PNC, SC105, SC205, SC50A, SC50B* an upper one although the corresponding distance to infeasibility is equal to 0. We guess that for these problems the distances to infeasibility given in the paper of Ordóñez and Freund [20] are in fact not zero. This may result from computing them numerically without verification.

The large relative errors μ for the problems *SCSD6, SCTAP2, SCTAP3* are due to the bad upper bounds. We hope to improve this in the future.

Table II shows that although almost all problems have unbounded variables, in many cases the rigorous lower bound can be computed within a fraction of the computational work that is required for solving the problem approximately. There $i_{\underline{f^*}}$, $i_{\overline{f^*}}$ denote the number of iterations for computing the lower and upper bound respectively and $t_{\underline{f^*}}/t_{f^*}$, $t_{\overline{f^*}}/t_{f^*}$ denote the corresponding time ratios. If this ratio can not be computed due to problems being solved faster than the timer resolution of 0.01 sec, this column is left empty. Only the problems *FIT1D, FIT2D, SIERRA* have finite simple bounds yielding an infinite distance to dual infeasibility. This results in a lower bound without the need of iterating. The problem *RECIPE* also delivers a lower bound without any iterations, and 65 problems deliver a lower bound in only 1 iteration. The huge time ratios for problems *AGG, BEACONFD*, and *SCRS8* stem from lp_solve timing out after 24 hours of trying to solve a perturbed problem.

Infinite error bounds for the optimal value result from ill-posedness and are expressed by exceeding iteration counts, rank deficient constraint matrices, or in five cases, by numerical problems during the solution of perturbed linear programs. Table II shows that the determination of an infinite bound is very time consuming if the iteration count exceeds the set limit of 31.

Since PROFIL/BIAS does not support sparse structures, the memory usage increases dramatically when transforming the constraint matrices from lp_solve's sparse representation to PROFIL/BIAS's non-sparse one. The usage of non-sparse interval linear solvers adds to this effect. This is the reason why in some cases even few iterations result in large time ratios. In the future, we will enhance PROFIL/BIAS to support sparse structures.

We emphasize that the bounds depend drastically on the used lp-solver. Even in the case where we used the same solver but a previous version (lp_solve 3.2) the results got worse. Some rigorous lower bounds computed with lp_solve 3.2 are listed in Table III. We mention that in all cases the approximations computed by lp_solve are within the rigorous bounds.

To test the quality of the algorithms when using interval data, the problem parameters of the Netlib lp problems were multiplied by the interval $[1 - 10^{-6}, 1 + 10^{-6}]$ yielding a relative uncertainty. Table IV contains the error bounds for these problems with interval parameters, and Table V shows the performance. Compared with the radius $r = 1 \cdot 10^{-6}$ of the interval input data, the algorithms give in most cases very accurate worst case bounds. The huge time ratios for *AGG, BECONFD, FFFFF800* and *PILOT.WE* originate again from lp_solve timeouts while trying to solve perturbed problems.

5. Concluding Remarks

Computing rigorous error bounds for the optimal value of linear programming problems together with a certificate of the existence of optimal solutions has been shown to be possible in various areas of applications.

These bounds can also be used in global optimization and mixed-integer nonlinear programming whenever linear relaxations must be solved in branch and bound algorithms.

In future implementations we plan to incorporate several improvements like sparse structures in PROFIL/BIAS, other linear programming solvers, other deflation parameters, and a rigorous preprocessing of the data.

Appendix

A. Tables

Table I.: Rigorous bounds for the Netlib problems
ρ_d – distance to dual infeasibility, ρ_p – distance to primal infeasibility,
$\underline{f^*}$ – lower bound, $\overline{f^*}$ – upper bound, μ – relative accuracy

Name	ρ_d	ρ_p	$\underline{f^*}$	$\overline{f^*}$	μ
25FV47	0	0	$5.5018e+03$	∞	$8.5111e-08$
80BAU3B	0	0	$9.8722e+05$	$9.8722e+05$	$5.6653e-08$
ADLITTLE	0.051651	0	$2.2549e+05$	$2.2549e+05$	$3.6470e-08$
AFIRO	1.000000	0.397390	$-4.6475e+02$	$-4.6475e+02$	$2.0481e-08$
AGG2	0.771400	0	$-2.0239e+07$	∞	$2.0868e-08$
AGG3	0.771400	0	$1.0312e+07$	∞	$7.3998e-08$
AGG	0.771400	0	$-3.5992e+07$	∞	$2.7323e-08$
BANDM	0.000418	0	$-1.5863e+02$	∞	$7.0742e-08$
BEACONFD	0	0	$3.3592e+04$	∞	$9.9997e-09$
BLEND	0.040726	0.003541	$-3.0812e+01$	$-3.0812e+01$	$1.3560e-07$
BNL1	0.106400	0	$1.9776e+03$	∞	$7.2244e-08$
BNL2	0	0	$1.8112e+03$	∞	$2.0899e-08$
BORE3D	0.003539	0	$1.3731e+03$	∞	$1.3362e-08$
BRANDY	0	0	$-\infty$	∞	NaN
CAPRI	0.095510	0.000252	$2.6900e+03$	$2.6900e+03$	$1.6905e-07$
CYCLE	0	0	$-5.2264e+00$	∞	$1.4574e-08$
CZPROB	0.008807	0	$2.1852e+06$	∞	$1.0915e-08$
D2Q06C	0	0	$1.2278e+05$	∞	$4.5242e-08$
D6CUBE	2.000000	0	$3.1549e+02$	∞	$1.6796e-08$
DEGEN2	1.000000	0	$-1.4352e+03$	∞	$9.3150e-09$
E226	0	0	$-2.5865e+01$	$-2.5865e+01$	$9.1411e-08$
ETAMACRO	0.200000	0	$-7.5572e+02$	∞	$4.4004e-09$
FFFFF800	0.033046	0	$5.5568e+05$	∞	$4.1052e-08$
FINNIS	0	0	$-\infty$	$1.7279e+05$	$4.8378e-08$
FIT1D	∞	3.500000	$-9.1464e+03$	$-9.1464e+03$	$6.1900e-09$
FIT1P	0.437500	1.271887	$9.1464e+03$	$9.1464e+03$	$1.1418e-08$
FIT2D	∞	317.000000	$-6.8464e+04$	$-6.8464e+04$	$4.8472e-09$
FIT2P	1.000000	1.057333	$6.8464e+04$	$6.8464e+04$	$6.9205e-09$
GANGES	1.000000	0	$-1.0959e+05$	∞	$3.5123e-09$
GFRD-PNC	0.347032	0	$6.9022e+06$	$6.9022e+06$	$5.5746e-08$
GREENBEA	0	0	$-\infty$	∞	NaN
GREENBEB	0	0	$-\infty$	∞	NaN
GROW15	0.968073	0.572842	$-1.0687e+08$	$-1.0687e+08$	$3.5135e-09$
GROW22	0.968073	0.572842	$-1.6083e+08$	$-1.6083e+08$	$3.7475e-09$
GROW7	0.968073	0.572842	$-4.7788e+07$	$-4.7788e+07$	$3.6032e-09$
ISRAEL	0.166850	0.027248	$-8.9664e+05$	$-8.9664e+05$	$1.5935e-08$
KB2	0.018802	0.000201	$-1.7499e+03$	$-1.7499e+03$	$2.1792e-08$
LOTFI	0	0.000306	$-\infty$	$-2.5265e+01$	$4.5049e-09$
MAROS	0	0	$-\infty$	∞	NaN
MAROS-R7	0.628096	1.000000	$1.4972e+06$	$1.4972e+06$	$8.5236e-09$
MODSZK1	0.108469	0	$3.2057e+02$	∞	$1.5512e-04$
PEROLD	0.000943	0	$-9.3808e+03$	∞	$2.2012e-08$
PILOT4	0.000075	0	$-2.5811e+03$	∞	$2.8098e-08$
PILOT87	0	0	$-\infty$	∞	NaN

continued...

Name	ρ_d	ρ_p	$\underline{f^*}$	$\overline{f^*}$	μ
PILOT.JA	0.000750	0	$-6.1131e+03$	∞	$1.5904e-08$
PILOTNOV	0.000750	0	$-4.4973e+03$	∞	$3.2619e-08$
PILOT.WE	0.044874	0	$-2.7201e+06$	∞	$5.2748e-08$
QAP8	4.000000	0	$2.0350e+02$	∞	$5.2913e-08$
RECIPE	0	0	$-2.6662e+02$	∞	$4.2641e-16$
SC105	0.133484	0	$-5.2202e+01$	$-5.2202e+01$	$7.7626e-08$
SC205	0.010023	0	$-5.2202e+01$	$-5.2202e+01$	$9.0740e-08$
SC50A	0.562500	0	$-6.4575e+01$	$-6.4575e+01$	$5.6764e-08$
SC50B	0.421875	0	$-7.0000e+01$	$-7.0000e+01$	$5.7599e-08$
SCAGR25	0.034646	0.021077	$-1.4753e+07$	$-1.4753e+07$	$3.7821e-08$
SCAGR7	0.034646	0.022644	$-2.3314e+06$	$-2.3314e+06$	$3.9152e-08$
SCFXM1	0	0	$-\infty$	∞	NaN
SCFXM2	0	0	$-\infty$	∞	NaN
SCFXM3	0	0	$-\infty$	∞	NaN
SCORPION	0.949393	0	$1.8781e+03$	∞	$2.7948e-08$
SCRS8	0	0	$9.0430e+02$	∞	$3.4248e-08$
SCSD1	1.000000	5.037757	$8.6667e+00$	$8.6668e+00$	$1.0579e-05$
SCSD6	1.000000	1.603351	$5.0500e+01$	$5.0707e+01$	$4.0917e-03$
SCSD8	1.000000	0.268363	$9.0500e+02$	∞	$6.3831e-08$
SCTAP1	1.000000	0.032258	$1.4122e+03$	∞	$2.1640e-08$
SCTAP2	1.000000	0.586563	$1.7248e+03$	$1.9777e+03$	$1.3662e-01$
SCTAP3	1.000000	0.381250	$1.4240e+03$	$2.0866e+03$	$3.7748e-01$
SHARE1B	0.000751	0.000015	$-7.6589e+04$	$-7.6589e+04$	$1.7119e-07$
SHARE2B	0.287893	0.001747	$-4.1573e+02$	$-4.1573e+02$	$4.0674e-07$
SHELL	1.777778	0	$1.2088e+09$	∞	$4.6203e-09$
SHIP04L	13.146000	0	$1.7933e+06$	∞	$9.7665e-09$
SHIP04S	13.146000	0	$1.7987e+06$	∞	$1.0115e-08$
SHIP08L	21.210000	0	$1.9091e+06$	∞	$1.0593e-08$
SHIP08S	21.210000	0	$1.9201e+06$	∞	$1.1197e-08$
SHIP12L	7.434000	0	$1.4702e+06$	∞	$1.1950e-08$
SHIP12S	7.434000	0	$1.4892e+06$	∞	$1.3700e-08$
SIERRA	∞	0	$1.5394e+07$	∞	$5.3601e-14$
STAIR	0	0.000580	$-\infty$	$-2.5127e+02$	$5.4796e-09$
STANDATA	1.000000	0	$1.2577e+03$	∞	$1.2619e-08$
STANDGUB	0	0	$1.2577e+03$	∞	$1.2619e-08$
STANDMPS	1.000000	0	$1.4060e+03$	∞	$1.3776e-08$
STOCFOR1	0.011936	0.001203	$-4.1132e+04$	$-4.1132e+04$	$4.2148e-08$
STOCFOR2	0.000064	0.000437	$-3.9024e+04$	$-3.9024e+04$	$5.6996e-08$
TRUSS	10.000000	0.518928	$4.5882e+05$	$4.5882e+05$	$2.3769e-06$
TUFF	0.017485	0	$2.8677e-01$	∞	$5.3744e-03$
VTP.BASE	0.500000	0	$1.2983e+05$	∞	$3.4508e-08$
WOOD1P	1.000000	0	$1.4429e+00$	∞	$4.3361e-08$
WOODW	1.000000	0	$1.3045e+00$	∞	$2.4401e-08$

Table II.: Performance of the Netlib bounds
$i_{\underline{f^*}}$ – iterations to compute lower bound,
$t_{\underline{f^*}}$ – time to compute lower bound, t_{f^*} – time to compute approximate solution,
$i_{\overline{f^*}}$ – iterations to compute upper bound,
$t_{\overline{f^*}}$ –time to compute upper bound

Name	$i_{\underline{f^*}}$	$t_{\underline{f^*}}/t_{f^*}$	$i_{\overline{f^*}}$	$t_{\overline{f^*}}/t_{f^*}$
25FV47	1	0.102	0	0.030
80BAU3B	1	1.943	3	0.808
ADLITTLE	1	0.000	4	0.000
AFIRO	1		5	
AGG2	1	1.000	31	7.667
AGG3	1	0.750	31	5.750
AGG	1	0.500	2	4320000.000
BANDM	1	0.300	31	58.000
BEACONFD	1	1.000	12	8640100.000
BLEND	1	0.000	5	1.000
BNL1	1	0.519	0	3.444
BNL2	3	2.648	31	197.643
BORE3D	1	1.000	0	11.000
BRANDY	31	2.000	0	0.000
CAPRI	1	0.333	4	9.333
CYCLE	10	2.574	0	1.626
CZPROB	1	0.469	31	68.531
D2Q06C	1	0.277	31	26.499
D6CUBE	1	0.345	0	7.480
DEGEN2	1	0.152	0	1.273
E226	1	0.167	4	0.500
ETAMACRO	3	0.636	0	6.182

continued. . .

Name	$i_{\underline{f^*}}$	$t_{\underline{f^*}}/t_{f^*}$	$i_{\overline{f^*}}$	$t_{\overline{f^*}}/t_{f^*}$
FFFFF800	1	0.421	31	35.000
FINNIS	31	3.500	2	2.250
FIT1D	0	0.000	2	0.125
FIT1P	1	0.321	10	68.887
FIT2D	0	0.003	1	0.009
FIT2P	1	0.722	12	243.487
GANGES	1	0.841	31	620.455
GFRD-PNC	1	0.833	7	173.417
GREENBEA	31	0.773	0	9.911
GREENBEB	31	0.813	0	10.568
GROW15	1	0.051	6	4.966
GROW22	1	0.060	9	13.667
GROW7	1	0.000	10	4.667
ISRAEL	1	0.000	1	0.333
KB2	1		4	
LOTFI	31	2.000	5	5.500
MAROS	31	0.919	0	2.364
MAROS-R7	1	0.381	10	105.517
MODSZK1	1	1.235	0	18.118
PEROLD	4	0.573	31	16.183
PILOT4	3	0.680	31	22.080
PILOT87	9	674.546	2	979.242
PILOT.JA	3	0.330	0	0.378
PILOTNOV	1	0.436	0	1.154
PILOT.WE	1	0.141	6	9.559
QAP8	10	21.895	0	0.091
RECIPE	0	0.000	0	0.000
SC105	1	0.000	1	0.000
SC205	1	0.500	1	2.000
SC50A	1		1	
SC50B	1	0.000	1	0.000
SCAGR25	1	0.308	4	27.769
SCAGR7	1	0.000	4	4.000
SCFXM1	31	1.667	31	16.500
SCFXM2	31	1.000	31	25.103
SCFXM3	31	0.909	31	32.212
SCORPION	1	0.500	0	38.250
SCRS8	1	1.000	7	785536.364
SCSD1	1	0.500	13	12.500
SCSD6	1	0.571	15	16.429
SCSD8	1	0.654	20	42.904
SCTAP1	1	0.500	31	18.750
SCTAP2	1	1.310	28	124.517
SCTAP3	1	2.500	30	174.761
SHARE1B	1	0.500	8	5.000
SHARE2B	1	0.000	5	1.000
SHELL	1	0.941	0	40.706
SHIP04L	1	0.938	0	0.438
SHIP04S	1	1.000	0	0.333
SHIP08L	1	1.089	0	0.500
SHIP08S	1	1.259	0	0.556
SHIP12L	1	1.203	0	0.576
SHIP12S	1	1.102	0	0.475
SIERRA	0	0.480	0	14.960
STAIR	31	3.900	1	5.900
STANDATA	1	2.000	31	43.000
STANDGUB	1	8.000	0	2.000
STANDMPS	1	1.000	31	39.778
STOCFOR1	1	0.000	10	8.000
STOCFOR2	1	1.000	13	189.336
TRUSS	1	0.236	15	35.163
TUFF	14	4.286	0	0.286
VTP.BASE	1	1.000	31	17.000
WOOD1P	1	0.302	31	22.698
WOODW	1	0.602	7	61.329

Table III.: Rigorous bounds for the Netlib problems using lp_solve 3.2
ρ_d – distance to dual infeasibility, ρ_p – distance to primal infeasibility,
$\underline{f^*}$ – lower bound, $\overline{f^*}$ – upper bound, μ – relative accuracy

Name	ρ_d	ρ_p	$\underline{f^*}$	$\overline{f^*}$	μ
SC105	0.133484	0	$-2.1696e+13$	$-5.2201e+01$	$1.0000e-00$
SC205	0.010023	0	$-\infty$	$-5.2201e+01$	$1.8125e-05$
SC50A	0.562500	0	$-5.8365e+04$	$-6.4574e+01$	$9.9889e-01$
SC50B	0.421875	0	$-7.3733e+02$	$-6.9999e+01$	$9.0506e-01$

Table IV.: Bounds for interval problems
ρ_d – distance to dual infeasibility, ρ_p – distance to primal infeasibility,
$\underline{f^*}$ – lower bound, $\overline{f^*}$ – upper bound, μ – relative accuracy

Name	ρ_d	ρ_p	$\underline{f^*}$	$\overline{f^*}$	μ
25FV47	0	0	$5.5013e+03$	∞	$9.2351e-05$
80BAU3B	0	0	$9.8720e+05$	$9.8726e+05$	$6.1059e-05$
ADLITTLE	0.051651	0	$2.2549e+05$	∞	$3.0616e-05$
AFIRO	1.000000	0.397390	$-4.6476e+02$	$-4.6460e+02$	$3.4213e-04$
AGG2	0.771400	0	$-2.0240e+07$	∞	$2.7134e-05$
AGG3	0.771400	0	$1.0311e+07$	∞	$9.2632e-05$
AGG	0.771400	0	$-3.5993e+07$	∞	$3.3187e-05$
BANDM	0.000418	0	$-1.5864e+02$	∞	$7.2054e-05$
BEACONFD	0	0	$3.3592e+04$	∞	$1.1010e-05$
BLEND	0.040726	0.003541	$-3.0816e+01$	$-3.0803e+01$	$4.2114e-04$
BNL1	0.106400	0	$1.9775e+03$	∞	$8.6913e-05$
BNL2	0	0	$1.8112e+03$	∞	$2.2986e-05$
BORE3D	0.003539	0	$1.3731e+03$	∞	$2.1177e-05$
BRANDY	0	0	$-\infty$	∞	NaN
CAPRI	0.095510	0.000252	$2.6895e+03$	$2.6935e+03$	$1.4821e-03$
CYCLE	0	0	$-\infty$	∞	NaN
CZPROB	0.008807	0	$2.1852e+06$	∞	$1.3316e-05$
D2Q06C	0	0	$1.2278e+05$	∞	$4.8916e-05$
D6CUBE	2.000000	0	$3.1549e+02$	∞	$2.0404e-05$
DEGEN2	1.000000	0	$-1.4352e+03$	∞	$1.0747e-05$
E226	0	0	$-1.8753e+01$	∞	$3.1879e-01$
ETAMACRO	0.200000	0	$-7.5573e+02$	∞	$2.0841e-05$
FFFFF800	0.033046	0	$5.5565e+05$	∞	$4.7258e-05$
FINNIS	0	0	$-\infty$	∞	NaN
FIT1D	∞	3.500000	$-9.1464e+03$	$-9.1462e+03$	$1.8396e-05$
FIT1P	0.437500	1.271887	$9.1463e+03$	$9.1476e+03$	$1.4385e-04$
FIT2D	∞	317.000000	$-6.8465e+04$	$-6.8463e+04$	$1.5199e-05$
FIT2P	1.000000	1.057333	$6.8464e+04$	$6.8469e+04$	$8.6123e-05$
GANGES	1.000000	0	$-1.0960e+05$	∞	$1.6019e-04$
GFRD-PNC	0.347032	0	$6.9018e+06$	∞	$5.7493e-05$
GREENBEA	0	0	$-\infty$	∞	NaN
GREENBEB	0	0	$-\infty$	∞	NaN
GROW15	0.968073	0.572842	$-1.0687e+08$	$-1.0687e+08$	$1.6424e-05$
GROW22	0.968073	0.572842	$-1.6084e+08$	$-1.6083e+08$	$1.7269e-05$
GROW7	0.968073	0.572842	$-4.7788e+07$	$-4.7787e+07$	$1.6718e-05$
ISRAEL	0.166850	0.027248	$-8.9665e+05$	$-8.9664e+05$	$1.8453e-05$
KB2	0.018802	0.000201	$-1.7499e+03$	$-1.7498e+03$	$1.0083e-04$
LOTFI	0	0.000306	$-\infty$	$-2.5254e+01$	$4.1965e-04$
MAROS	0	0	$-\infty$	∞	NaN
MAROS-R7	0.628096	1.000000	$1.4972e+06$	$1.4973e+06$	$6.3798e-05$
MODSZK1	0.108469	0	$2.7040e+02$	∞	$1.6995e-01$
PEROLD	0.000943	0	$-\infty$	∞	NaN
PILOT4	0.000075	0	$-2.5813e+03$	∞	$7.4535e-05$
PILOT87	0	0	$3.0170e+02$	∞	$1.8799e-05$
PILOT.JA	0.000750	0	$-6.1134e+03$	∞	$5.1024e-05$
PILOTNOV	0.000750	0	$-4.4974e+03$	∞	$3.3681e-05$
PILOT.WE	0.044874	0	$-2.7204e+06$	∞	$1.2197e-04$
QAP8	4.000000	0	$-\infty$	∞	NaN
RECIPE	0	0	$-2.6662e+02$	∞	$2.3744e-05$
SC105	0.133484	0	$-5.2205e+01$	$-5.2199e+01$	$1.2454e-04$
SC205	0.010023	0	$-5.2206e+01$	∞	$7.3685e-05$
SC50A	0.562500	0	$-6.4578e+01$	$-6.4573e+01$	$7.5860e-05$
SC50B	0.421875	0	$-7.0003e+01$	$-6.9998e+01$	$7.7140e-05$
SCAGR25	0.034646	0.021077	$-1.4754e+07$	$-1.4752e+07$	$1.1701e-04$
SCAGR7	0.034646	0.022644	$-2.3315e+06$	$-2.3312e+06$	$1.2282e-04$
SCFXM1	0	0	$-\infty$	∞	NaN
SCFXM2	0	0	$-\infty$	∞	NaN
SCFXM3	0	0	$-\infty$	∞	NaN
SCORPION	0.949393	0	$1.8781e+03$	∞	$2.9037e-05$
SCRS8	0	0	$9.0426e+02$	∞	$3.7130e-05$
SCSD1	1.000000	5.037757	$8.6665e+00$	$8.6677e+00$	$1.4261e-04$
SCSD6	1.000000	1.603351	$5.0499e+01$	$5.0506e+01$	$1.4791e-04$
SCSD8	1.000000	0.268363	$9.0494e+02$	$9.0550e+02$	$6.2245e-04$
SCTAP1	1.000000	0.032258	$1.4122e+03$	$1.4125e+03$	$2.2764e-04$
SCTAP2	1.000000	0.586563	$1.7248e+03$	$1.7260e+03$	$6.8090e-04$
SCTAP3	1.000000	0.381250	$1.4240e+03$	∞	$1.1257e-05$
SHARE1B	0.000751	0.000015	$-7.6602e+04$	$-7.6428e+04$	$2.2704e-03$
SHARE2B	0.287893	0.001747	$-4.1584e+02$	$-4.1559e+02$	$6.1583e-04$
SHELL	1.777778	0	$1.2088e+09$	∞	$1.4568e-05$
SHIP04L	13.146000	0	$1.7933e+06$	∞	$1.0788e-05$
SHIP04S	13.146000	0	$1.7987e+06$	∞	$1.1168e-05$
SHIP08L	21.210000	0	$1.9090e+06$	∞	$1.1798e-05$

continued...

Name	ρ_d	ρ_p	$\underline{f^*}$	$\overline{f^*}$	μ
SHIP08S	21.210000	0	$1.9201e+06$	∞	$1.2431e-05$
SHIP12L	7.434000	0	$1.4702e+06$	∞	$1.3344e-05$
SHIP12S	7.434000	0	$1.4892e+06$	∞	$1.5090e-05$
SIERRA	∞	0	$1.5376e+07$	∞	$1.1848e-03$
STAIR	0	0.000580	$-\infty$	$-2.5126e+02$	$3.7820e-05$
STANDATA	1.000000	0	$1.2577e+03$	∞	$1.3912e-05$
STANDGUB	0	0	$1.2577e+03$	∞	$1.3912e-05$
STANDMPS	1.000000	0	$1.4060e+03$	∞	$1.4991e-05$
STOCFOR1	0.011936	0.001203	$-4.1134e+04$	$-4.1126e+04$	$1.8826e-04$
STOCFOR2	0.000064	0.000437	$-3.9026e+04$	∞	$4.4179e-05$
TRUSS	10.000000	0.518928	$4.5877e+05$	$4.5910e+05$	$7.1544e-04$
TUFF	0.017485	0	$2.9213e-01$	∞	$1.9790e-05$
VTP.BASE	0.500000	0	$1.2982e+05$	∞	$5.9034e-05$
WOOD1P	1.000000	0	$1.4428e+00$	∞	$4.7868e-05$
WOODW	1.000000	0	$1.3044e+00$	∞	$2.6084e-05$

Table V.: Performance of the interval bounds
$i_{\underline{f^*}}$ – iterations to compute lower bound,
$t_{\underline{f^*}}$ – time to compute lower bound, t_{f^*} – time to compute approximate solution,
$i_{\overline{f^*}}$ – iterations to compute upper bound,
$t_{\overline{f^*}}$ –time to compute upper bound

Name	$i_{\underline{f^*}}$	$t_{\underline{f^*}}/t_{f^*}$	$i_{\overline{f^*}}$	$t_{\overline{f^*}}/t_{f^*}$
25FV47	1	0.107	0	0.030
80BAU3B	1	1.921	1	0.393
ADLITTLE	1		31	
AFIRO	1		9	
AGG2	1	0.750	31	3.250
AGG3	1	0.750	31	3.250
AGG	1	0.667	2	2880000.000
BANDM	1	0.273	31	14.455
BEACONFD	1	0.500	9	8640050.000
BLEND	1	0.000	9	1.000
BNL1	1	0.517	0	3.172
BNL2	1	1.809	31	87.787
BORE3D	1	0.333	0	6.667
BRANDY	31	1.333	0	0.333
CAPRI	3	1.000	8	7.250
CYCLE	19	11.365	0	1.670
CZPROB	1	0.420	31	40.407
D2Q06C	1	0.283	31	15.738
D6CUBE	1	0.360	0	7.500
DEGEN2	1	0.152	0	1.818
E226	1	0.200	31	4.000
ETAMACRO	1	0.417	0	5.750
FFFFF800	1	0.421	3	454752.632
FINNIS	31	2.625	2	1.250
FIT1D	0	0.000	5	0.250
FIT1P	1	0.340	2	35.887
FIT2D	0	0.005	4	0.028
FIT2P	1	0.728	2	158.333
GANGES	1	0.889	31	588.844
GFRD-PNC	1	0.917	9	153.417
GREENBEA	31	0.391	0	9.666
GREENBEB	31	0.434	0	10.398
GROW15	1	0.033	2	3.083
GROW22	1	0.059	2	6.812
GROW7	1	0.200	2	2.800
ISRAEL	1	0.500	1	0.000
KB2	1		4	
LOTFI	31	1.500	2	3.500
MAROS	31	0.505	0	2.218
MAROS-R7	1	0.404	2	64.692
MODSZK1	11	5.688	0	19.500
PEROLD	13	4.159	31	14.732
PILOT4	7	1.250	31	11.917
PILOT87	1	0.073	31	0.711
PILOT.JA	7	0.582	0	0.365
PILOTNOV	1	0.436	0	1.103
PILOT.WE	6	0.337	18	29697.938
QAP8	18	753.096	0	0.088
RECIPE	0		0	
SC105	1		6	

continued. . .

Name	$i_{\underline{f}^*}$	$t_{\underline{f}^*}/t_{f^*}$	$i_{\overline{f}^*}$	$t_{\overline{f}^*}/t_{f^*}$
SC205	1	0.500	31	8.500
SC50A	1		6	
SC50B	1		2	
SCAGR25	1	0.308	5	18.615
SCAGR7	1	0.000	3	3.000
SCFXM1	31	1.167	31	8.333
SCFXM2	31	0.621	31	12.655
SCFXM3	31	0.478	31	17.567
SCORPION	1	0.500	0	26.750
SCRS8	1	1.000	31	104.000
SCSD1	1	0.500	12	11.500
SCSD6	1	0.714	12	14.857
SCSD8	1	4.635	14	33.923
SCTAP1	1	0.750	9	7.000
SCTAP2	1	1.407	9	65.370
SCTAP3	1	2.667	31	104.622
SHARE1B	1	0.500	8	4.000
SHARE2B	1	0.000	6	1.000
SHELL	1	1.125	0	62.500
SHIP04L	1	0.941	0	0.353
SHIP04S	1	0.917	0	0.333
SHIP08L	1	1.125	0	0.482
SHIP08S	1	1.214	0	0.536
SHIP12L	1	1.239	0	0.769
SHIP12S	1	1.049	0	0.508
SIERRA	0	0.480	0	14.320
STAIR	31	3.300	6	9.400
STANDATA	1	7.000	31	80.000
STANDGUB	1	4.000	0	1.000
STANDMPS	1	1.250	31	23.625
STOCFOR1	1	0.000	15	4.000
STOCFOR2	1	1.076	31	211.750
TRUSS	1	0.200	11	29.779
TUFF	12	4.286	0	0.429
VTP.BASE	1	0.500	1	0.500
WOOD1P	1	0.333	0	8.405
WOODW	1	0.531	3	46.261

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