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Letter to the Editor

A property of the nearly optimal root-bound

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Abstract

The importance of root-bounds for practical and theoretical algorithms for polynomial root-approximation is well-known. The root-bound by Fujiwara was shown to be near optimal by van der Sluis, and is the most often used in practice. We show here that this bound always compares favorably with Kojima's bound, a question left open in the work of van der Sluis.

Keywords: Complex polynomials; Root-bounds; Fujiwara's bound

The task of root-approximation is of importance in diverse fields as, e.g., control engineering, robotics and algorithm complexity. A practical bound should be near on average to the correct absolute value. For example, the perturbation estimates on the Hausdorff distance of two polynomials are expressed dependent on root-bounds, cf. [4]. This affects numerical validation of roots as well as the computational complexity estimates of path following [6]. The hybrid method for root-cluster verification suggested in Rump's recent study [5] starts out from inclusion estimates via root-bounds. Also the asymptotically nearly optimal root finding methods of Pan [4] employ this bound in its basic steps. Practical and theoretical needs are met with the Fujiwara bound [1] (stated below) of 1916; this was demonstrated by van der Sluis in [7]. However, no direct comparison has yet been given to the related bound of Kojima [2]. The comparison of the Kojima bound (designated by T in [7]) with the Fujiwara bound (designated by S op.cit.) on the class of all polynomials with respect to the maximum overestimation factor *mof* was concluded in [7] as follows:

"Initially, one could have hoped that T [*the Kojima bound*], though not always defined, might do well just in those cases where S [*the Fujiwara bound*] gives a severe overestimation".

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We will show in the following that the Fujiwara bound always compares favorably with the bound of Kojima. This fact is a consequence of interpreting the terms involved in the bounds as certain means.

Definition and Lemma. *Given* $p(z) = \sum_{i=0}^{n} a_i z^i$, $a_n \neq 0$. *Define*

$$F(p) := 2\max\left\{\left|\frac{a_{n-1}}{a_n}\right|, \left|\frac{a_{n-2}}{a_n}\right|^{1/2}, \ldots, \left|\frac{a_1}{a_n}\right|^{1/(n-1)}, \left|\frac{a_0}{2a_n}\right|^{1/n}\right\}. \tag{1}$$

Whenever $a_{n-1}\,a_{n-2}\,\cdots\,a_1 \neq 0$, *set*

$$K(p) := 2\max\left\{\left|\frac{a_{n-1}}{a_n}\right|, \left|\frac{a_{n-2}}{a_{n-1}}\right|, \ldots, \frac{|a_1|}{|a_2|}, \frac{1}{2}\frac{|a_0|}{|a_1|}\right\}. \tag{2}$$

Then $\mu(p) := \max\{|\zeta| : p(\zeta) = 0\}$ is bounded by

$$\mu(p) \leqslant F(p), \quad \text{and} \quad \mu(p) \leqslant K(p).$$

(Proofs and references may be found in [3].)

Both root-bounds may yield eventually the same, sharp bound [7].

$$g(z) := z^n - z^{n-1} - \cdots - z - 2 = (z-2)(z^n - 1)/(z-1)$$

$$F(g) = K(g) = 2. \tag{3}$$

Theorem 1. *Given* $p(z) = \sum_{i=0}^{n} a_i z^i \in \mathbb{C}[z]$ *with* $a_1\,a_2\,\cdots\,a_n \neq 0$. *Then* $F(p) \leqslant K(p)$.

Proof. The quotient terms involved in Kojima's bound [2] are well defined. Let us consider now the successive geometric means of these quotient terms which are

$$\left\{\left|\frac{a_{n-1}}{a_n}\right|, \left|\frac{a_{n-2}}{a_n}\right|^{1/2}, \ldots, \left|\frac{a_1}{a_n}\right|^{1/(n-1)}, \left|\frac{a_0}{2a_n}\right|^{1/n}\right\}. \tag{4}$$

The maximum of these successive geometric means multiplied by 2 is precisely Fujiwara's bound $F(p)$, which is hence no larger than Kojima's bound $K(p)$. □

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