

On upper bounds for real proportional stabilizing controllers

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Abstract

The explicit determination of the largest admissible constant controller is difficult. For linear systems with all poles in the domain of stability but at least one non-minimum phase zero outside, an improved upper bound on real proportional gain controllers is given. Examples show the improvement and further possible margins.¹

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1 Introduction

Given the transfer function of a plant $p(s)$ and a controller $c(s)$ stabilizing it in closed loop. What conditions are imposed by the plant configuration on the norm of the controller?

Designate by $S(s) = (1 + p(s) \cdot c(s))^{-1}$ the so-called sensitivity function. It is then well-known that asymptotic internal stability of the closed feedback loop is equivalent to the following three properties [7] :

- i) $S(s)$ has poles just inside the domain of stability (i.e., the open left half-plane).
- ii) Zeros $(S(s)) \supset$ Poles $(p(s))$ (outside the domain of stability).
- iii) Zeros $(S(s) - 1) \supset$ Zeros $(p(s))$ (outside the domain of stability).

In this presentation, we study the following question: How large may we take a stabilizing proportional controller? We will exhibit a new upper bound (cf. Cor. 1) for rational (or more general: meromorphic) systems with all poles in the left half-plane,

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but at least one non-minimum phase zero in the right half-plane. The bound is related to the sensitivity of zeros. It does not depend on the number of zeros (comp. Th. 1).

We will outline analytic limitations to proportional controllers in the next section, present the steps leading to the bound in the following section, give the proofs subsequently, and close the paper with two worked examples.

2 Known limitation for proportional control

Consider the closed right half-plane and suppose the function $p(s)$ has no poles there.

If we consider only scalar, real controllers $c(z) = k$, we find that of conditions i), ii), iii) as above only i) means a limitation. For $S(z) = (1 + p(z) \cdot c(z))^{-1}$ to have no poles in the right half-plane, while $c(z) = k$, this means $\frac{-1}{p(z)} \neq k$. This yields the following.

Fact: *Every real controller $c(z) = k$ with*

$$|k| < [\sup \{|p(z)| : \Re z \geq 0, \Im(p(z)) = 0\}]^{-1} \quad (1)$$

is stabilizing.

The right hand side of (1) was denoted L_{opt} by Blondel and Bertilsson in [3] as this figure is the maximal size of the largest proportional gain controller such that all smaller ones are stabilizing. This value L_{opt} might in practice often be replaced by the unconstrained figure $\underline{L} := [\sup \{|p(z)| : \Re z \geq 0\}]^{-1}$ as $\underline{L} \leq L_{opt}$ is easier to compute. To assess any such approximation of the optimum upper bounds may be used. Thus, Blondel and Bertilsson [3] proved the following upper bound on *complex* proportional controllers k .

Theorem 1 *Let $p(s) = \sum_{v=0}^{\infty} p_v z^v$ be the transfer function of a system with no poles in the closed right-half plane. Suppose $p(s)$ has at least one zero s_0 in the open right half-plane. Let m designate the number of zeros of $p(s)$ in the open right half plane, q the multiplicity of the zero s_0 and $\Re(s_0)$ the real part of the zero s_0 . Suppose $k \in \mathbb{C}$ to be a stabilizing proportional controller. Then ²*

$$|k| < \frac{9 \cdot (m+1) q!}{|2 \cdot \Re(s_0)|^q \cdot |p^{(q)}(s_0)|}. \quad (2)$$

The above bound holds a fortiori for real k . The proof in [3] relies on the study of functions omitting two values. As the number of zeros is supposed to be finite and known, a complex function omitting the values 1 and 0 is constructed. Using the sharp version of a special result by Landau and Carathéodory (see [5]), the above bound is established. The quantity on the right hand side of (2) is denoted by \overline{U} in [3], and is such that no complex controller of larger absolute value is stabilizing. Hence, we have

$$\underline{L} \leq L_{opt} \leq \overline{U}. \quad (3)$$

²Please note: The term $q!$ is missing in [3].

We might want to inquire which real proportional controllers are not stabilizing the system. The general bound for complex proportional gain may be too conservative for this important case. Hence, we consider the largest absolute value R such that all *real* proportional compensators smaller in absolute value are stabilizing. In the following, we establish a bound $\bar{R} > R$ for *real* gain controllers with

$$\underline{L} \leq L_{opt} \leq \bar{R} < 9\bar{R} \leq \bar{U}.$$

In general, \bar{R} compares to \bar{U} as the constant 2 to the factor $9(m+1)$ in (2).

The improvement is achieved by studying the maximal possible set of image values. This value set must be bounded for the functions considered. Taking into consideration the image covered by $p(z)$ depending on the maximum real value, we may establish a new bound using ‘standard’ conformal mappings to the unit disc. As we will see, we can sharpen this further to obtain bounds near the order of magnitude of the optimal value L_{opt} .

3 Functions with image restrictions

We build our analysis on the simple fact that the coefficients of a bounded function are bounded. One of the quantitative expressions for this is Gutzmer’s lemma (see for ex. [6]), a consequence of the maximum principle. We denote the open unit disc by \mathbb{D} .

Lemma 1 (*Gutzmer*)

Suppose $g(z) : \mathbb{D} \rightarrow \mathbb{D}$ given as $g(z) = \sum_{v=0}^{\infty} g_v z^v$ is holomorphic. Then

$$\sum_{v=0}^{\infty} |g_v|^2 \leq 1.$$

We will derive the new bound as a consequence of the following observation, which is proved together with the theorem in the next section.

Proposition 1 Suppose we have a meromorphic function $p(z) = \sum_{v=0}^{\infty} p_v z^v$ with no poles in the closed unit disc $\bar{\mathbb{D}}$, but a zero z_0 at the origin of arbitrary multiplicity $q \geq 1$. Define $\delta := \sup \{|p(z)| : z \in \bar{\mathbb{D}}, \Im(p(z)) = 0\}$.

Then

$$\frac{1}{2}|p_q| = \frac{1}{2} \left| \frac{p^{(q)}(0)}{q!} \right| \leq \delta.$$

Moreover, for a simple root

$$\left| \frac{p_1}{2\delta} \right|^2 + \left| \frac{p_2}{2\delta} \right|^2 \leq 1.$$

For the half-plane $\Re s > 0$ and an arbitrary zero there, the proposition translates via conformal mapping to the following.

Theorem 2 Given a meromorphic transfer function $p(s)$ with no poles in the closed right half-plane. Define $\kappa := \sup\{|p(s)| : \Re s \geq 0, \text{Im}(p(s)) = 0\}$.

Suppose that $s_0, \Re s_0 > 0$ is a zero of $p(s)$ with multiplicity q . Then

$$\frac{1}{2} |2 \cdot \Re(s_0)|^q \cdot |p^{(q)}(s_0)/q!| \leq \kappa. \quad (4)$$

With

$$f_1 := -2\Re s_0 |p'(s_0)|,$$

$$f_2 := 2 \left(\frac{s_0 + 1}{s_0 + 1} \right)^2 \left(p''(s_0) (\Re s_0)^2 - \frac{p'(s_0) \Re s_0}{s_0 + 2} \right),$$

for a simple root s_0 the bound κ is no less than the smallest positive root of

$$\left| \frac{f_1}{2\kappa} \right|^2 + \left| \frac{f_2}{2\kappa} \right|^2 = 1. \quad (5)$$

As it is clear from the above discussion of (1), the inverse of the maximum real value on the half-plane, i.e. $1/\kappa$, gives the supremum bound such that all smaller real proportional gains are stabilizing controllers, i.e. L_{opt} . We infer our desired bound \bar{R} for proportional controllers by taking the inverse of our function bound.

Corollary 1 Given $p(s)$ meromorphic, with no poles in the closed right half-plane and at least one zero s_0 in the open half-plane. The largest real value R such that all real k with $|k| < R$ are proportional stabilizing $p(s)$ is bounded by

$$R < \frac{2}{|2 \cdot \Re(s_0)|^q \cdot |p^{(q)}(s_0)/q!|} =: \bar{R}, \quad (6)$$

where q denotes the multiplicity of the zero. Moreover, for the case of a simple root an upper bound \bar{R} is the inverse of the smallest positive root of (5).

Remark. Using the full force of Lemma 1 or related results, computing more coefficients we might improve this further. We give no explicit formulas here.

4 Derivation of results

Proof of the Proposition:

The meromorphic function $p(z)$ has by assumption no poles in the disc $|z| \leq 1$. The function values taken on the unit disc hence lie in a bounded domain. Thus, the limes superior of all real values, δ , is finite. Therefore, the image of $r(z)$ is contained in the doubly slit plane $M := \mathbb{C} \setminus \{(-\infty, -\delta] \cup [\delta, \infty)\}$.

We map this maximal image domain M back to the unit circle \mathbb{D} via

$$\phi(s) = \frac{1 - \sqrt{\frac{1-s/\delta}{1+s/\delta}}}{1 + \sqrt{\frac{1-s/\delta}{1+s/\delta}}}, \quad (7)$$

where the branch of the square-root is chosen such that $\sqrt{1} = 1$.

Obviously, $\phi(0) = 0$. We have $\phi(M) \subset \mathbb{D}$, especially

$$\phi(p(\mathbb{D})) \subset \mathbb{D}. \quad (8)$$

Let the power series expansion of $p(z)$ be given as: $p(z) = \sum_{v=0}^{\infty} p_v z^v$. Consider the composite function $\phi(p(z))$, and compute the first coefficients of the square root term in (7)

$$\begin{aligned} \sqrt{\frac{1-p(z)/\delta}{1+p(z)/\delta}} &= \sqrt{1 - \frac{2}{\delta} \frac{p(z)}{1 + \frac{p(z)}{\delta}}} \\ &\stackrel{p(0)=0}{=} \sqrt{1 - \frac{2}{\delta} \cdot \frac{(p_1 \cdot z + p_2 z^2 + \dots)}{1 + (1/\delta)(p_1 z + p_2 z^2 + \dots)}} \\ &= 1 - \frac{p_1}{\delta} \cdot z + \left(\frac{1}{2} \frac{p_1^2}{\delta^2} - \frac{p_2}{\delta} \right) z^2 + z^3 \cdot (\dots). \end{aligned}$$

Thus, if we consider our mapping $\phi(p(z))$ we have

$$\begin{aligned} \phi(p(z)) &= \frac{1 - \left(1 - \frac{p_1 z}{\delta} + \left[\frac{1}{2} \frac{p_1^2}{\delta^2} - \frac{p_2}{\delta} \right] \cdot z^2 + \dots \right)}{1 + \left(1 - \frac{p_1}{\delta} \cdot z + \left[\frac{1}{2} \frac{p_1^2}{\delta^2} - \frac{p_2}{\delta} \right] \cdot z^2 \dots \right)} \\ &= \frac{p_1}{2\delta} \cdot z + \left(\frac{p_2}{2\delta} \right) \cdot z^2 + \dots \end{aligned}$$

Using now Gutzmer's coefficient bound for the first coefficient $\phi(p(z))$, we find $\left| \frac{1}{2\delta} p_1 \right|^2 \leq 1$, hence $\left| \frac{p_1}{2} \right| \leq \delta$. We observe, that the analogue result holds true for p_q in place of p_1 , if $0 = p_0 = p_1 = \dots = p_{q-1}$. The first claim of the proposition follows as $p_q = p^{(q)}(0)/q!$.

Consider the special case of a simple zero. Using Gutzmer's bound again, we find

$$\left| \frac{p_1}{2\delta} \right|^2 + \left| \frac{p_2}{2\delta} \right|^2 \leq 1$$

□

Proof of the Theorem:

We proceed as in [3], to compose a suitable mapping of the unit disc to the right-half plane which transfers the origin to the zero s_0 .

For the canonical mapping of the unit disc to the right half-plane $H^+ := \{z : \Re z > 0\}$,

$$\sigma: \mathbb{D} \rightarrow H^+, \omega \mapsto \frac{1+\omega}{1-\omega}$$

define z_0 by

$$\sigma(z_0) = s_0 \Leftrightarrow \frac{1+z_0}{1-z_0} = s_0 \Leftrightarrow z_0 = \frac{s_0-1}{s_0+1} = 1 - \frac{2}{s_0+1}.$$

Thus, given z_0 we define

$$\mu: \mathbb{D} \rightarrow \mathbb{D}, \lambda \mapsto \frac{\lambda - z_0}{\lambda \cdot \bar{z}_0 - 1},$$

and the composite function $\sigma(\mu(\cdot))$ maps the unit disc to the right half plane. As $\mu(0) = z_0$ and $\sigma(z_0) = s_0$, the origin is mapped to s_0 .

As by assumption, s_0 is a q -fold zero of $p(s)$, there is a q -fold zero of $f(z) := p(\sigma(\mu(z))) = \sum f_\nu z^\nu$ at the origin. Using our proposition, we find that the real maximum δ of f on the unit circle which is equal the real maximum κ of p on the half-plane is lower bounded by

$$\left| \frac{f_q}{2} \right| \leq \kappa.$$

For a simple zero we obtain

$$\left| \frac{f_1}{2\kappa} \right|^2 + \left| \frac{f_2}{2\kappa} \right|^2 \leq 1. \quad (9)$$

This leaves us the task to compute f_1, f_2 for simple zeros in terms of our original function $p(s)$. (The case of a multiple zero is completely analogue).

We find for

$$f(z) = p(\sigma(\mu(z)))$$

that

$$f'(z) = p'(\sigma(\mu(z))) \cdot \sigma'(\mu(z)) \cdot \mu'(z) \quad (10)$$

and

$$\begin{aligned} f''(z) = & p''(\sigma(\mu(z))) \cdot (\sigma'(\mu(z)) \cdot \mu'(z))^2 + \\ & \dots + p'(\sigma(\mu(z))) \left[\sigma''(\mu(z)) (\mu'(z))^2 + \sigma'(\mu(z)) \mu''(z) \right] \end{aligned} \quad (11)$$

From the definition, we have

$$\begin{aligned}
\sigma(z) &= \frac{1+z}{1-z} = 1 + \frac{2z}{(1-z)}, \sigma'(z) = \frac{2(1-z) + 2z}{(1-z)^2} = \frac{2}{(1-z)^2} \\
\sigma''(z) &= \frac{-2(-2)(1-z)}{(1-z)^4} = \frac{4}{(1-z)^3} = \frac{2\sigma'(z)}{(1-z)}, \\
\mu(z) &= \frac{z-z_0}{z \cdot \bar{z}_0 - 1}, \\
\mu'(z) &= \frac{(z \cdot \bar{z}_0 - 1) - \bar{z}_0 \cdot (z - z_0)}{(z \cdot \bar{z}_0 - 1)^2} \\
&= \frac{z(\bar{z}_0 - \bar{z}_0) + z_0 \cdot \bar{z}_0 - 1}{(z \cdot \bar{z}_0 - 1)^2} = \frac{z_0 \cdot \bar{z}_0 - 1}{(z \cdot \bar{z}_0 - 1)^2}, \\
\mu''(z) &= -2\bar{z}_0(z \cdot \bar{z}_0 - 1) \cdot (z_0 \cdot \bar{z}_0 - 1) / (z \cdot \bar{z}_0 - 1)^4, \\
\mu'(0) &= z_0 \cdot \bar{z}_0 - 1, \\
\mu''(0) &= 2 \cdot \bar{z}_0 \cdot (z_0 \cdot \bar{z}_0 - 1) = 2 \cdot \bar{z}_0 \cdot (\mu'(0)).
\end{aligned}$$

This gives the first coefficient of f by (10) as

$$\begin{aligned}
f_1 = f'(0) &= p'(s_0) \cdot \sigma'(z_0) \cdot \mu'(0) \\
&= p'(s_0) \cdot \frac{2}{(1-z_0)^2} \cdot (z_0 \cdot \bar{z}_0 - 1).
\end{aligned}$$

With $z_0 = \sigma(s_0) = \frac{s_0-1}{s_0+1} = 1 - \frac{2}{s_0+1}$ we find

$$f_1 = -2 p'(s_0) \cdot \Re(s_0) \cdot \frac{s_0+1}{s_0+1}. \quad (12)$$

With s_0 inside the right half-plane the absolute value of this coefficient is:

$$|f_1| = |p'(s_0)| \cdot 2 \cdot \Re(s_0)$$

Computing the derivatives from (10), this translates (just as in [3]) to the situation of a multiple zero as:

$$|f_q| = |p^{(q)}(s_0)/q!| \cdot |2 \cdot \Re(s_0)|^q$$

Suppose now finally the case of a simple zero. The computation of f_2 from (11) is slightly more tedious. We have

$$\begin{aligned}
& p''(\sigma(\mu(z))) \cdot (\sigma'(\mu(z)) \cdot \mu'(z)) \Big|_{z=0} \\
&= p''(s_0) \cdot \left(-2 \cdot \Re(s_0) \cdot \frac{s_0+1}{s_0+1} \right)^2.
\end{aligned}$$

The term $\left[\sigma''(\mu(z)) \cdot (\mu'(z))^2 + \sigma'(\mu(z)) \cdot \mu''(z) \right]$ evaluated at zero gives

$$\begin{aligned} & \sigma''(\mu(0)) \cdot (\mu'(0))^2 + \sigma'(\mu(0)) \cdot \mu''(0) \\ &= \sigma'(\mu(0)) \cdot (\mu'(0)) \left[\frac{2 \cdot \mu'(0)}{1 - \mu(0)} + 2\bar{z}_0 \right]. \end{aligned}$$

The term $\left[\frac{2 \cdot \mu'(0)}{1 - \mu(0)} + 2\bar{z}_0 \right]$ where $\mu'(0) = z_0 \cdot \bar{z}_0 - 1$, $\mu(0) = z_0$ gives

$$\begin{aligned} & 2 \cdot \frac{(z_0 \cdot \bar{z}_0 - 1)}{1 - z_0} + 2 \cdot \bar{z}_0 \frac{(1 - z_0)}{1 - z_0} \\ &= -2 \cdot \frac{(\bar{z}_0 - 1)}{z_0 - 1} = 2 \cdot \frac{s_0 + 1}{s_0 + 1}. \end{aligned}$$

The second coefficient $f_2 = f''(0)/2$ finally computed as

$$f_2 = 2 \left(\frac{s_0 + 1}{s_0 + 1} \right)^2 (p''(s_0)(\Re s_0)^2 + p'(s_0)\Re s_0). \quad (13)$$

Hence, the lower bound for simple zeros is obtained from (9) with (12) and (13). \square

Remark. The Corollary may be strengthened by further coefficient computations. We may have given more formulas to state a further improved estimate for multiple zeros as well. The way to proceed is strictly as above.

5 Examples

Example 1.

In [3] the following transfer function was considered

$$p(s) = \frac{(s-2)(s+1)}{2s^3 + s^2 + 3s + 1}. \quad (14)$$

There is no pole in the closed right half plane and a single zero s_0 with $\Re s_0 > 0$, thus $s_0 = 2$, $m = 1$. The derivative of $p(z)$ is

$$p'(z) = \frac{-2 \cdot s^4 + 4 \cdot s^3 + 16 \cdot s^2 + 6s + 5}{(2 \cdot s^3 + s^2 + 3 \cdot s + 1)^2}$$

hence $p'(s_0) = p'(2) = 1/9$.

Blondel and Bertilsson derived the upper bound for the modulus of a *complex* gain as (2), namely

$$\frac{9 \cdot (m+1) q!}{|2\Re(s_0)|^q |p^q(s_0)|},$$

which evaluates to

$$\frac{9 \cdot 2}{4 \cdot 1/9} = 40.5.$$

From the first bound in Theorem 2 we find the upper bound \bar{R} to be

$$\frac{2}{f_1} = \frac{2}{|2 \cdot \operatorname{Re}(s_0)| \cdot |p'(s_0)|} = \frac{9}{2} = 4.5.$$

Willing to use more information from the transfer function we calculate first $p''(s_0) = \frac{-44}{243}$, and then

$$\begin{aligned} f_2 = f''(0)/2 &= 2 \left(\frac{3}{3}\right)^2 \left(p''(2)2^2 - p'(2)\frac{2}{4}\right) \\ &= 2\left(\frac{-44}{243} \cdot 4 + \frac{2}{9}\right) = \frac{-244}{243}. \end{aligned}$$

This gives by the second estimate of our Theorem 2 a lower bound of the maximal real value κ via

$$\left|\frac{f_1}{2\kappa}\right|^2 + \left|\frac{f_2}{2\kappa}\right|^2 \leq 1 \quad \text{as} \quad \frac{10}{243}\sqrt{178} \leq \kappa.$$

Taking the inverse gives the second controller bound of Corollary 1. Hence we find that the upper bound \bar{R} to real proportional gain is no greater than

$$1.8213.$$

What is the actual bound R for a constant real controller k ? Computing the root loci we find that the small gain controllers are confined to the interval

$$[-0.1625, 0.5].$$

Thus, we must compare the new upper bounds 4.5 and to 0.1625 which shows that further improvement of the latter bound is limited to a factor smaller than 13.95.

Example 2.

Consider the transfer function

$$\begin{aligned} p(s) &= \frac{s^2 - 10s + 37.25}{s^3 + 2.5s^2 + 3s + 1} \\ &= \frac{(s - (5 + 3.5 \cdot I))(s - (5 - 3.5 \cdot I))}{s^3 + 2.5s^2 + 3s + 1}. \end{aligned}$$

Evaluating the first and second derivative at the zero $5 + 3.5 \cdot I$ yields

$$\begin{aligned} p'(s_0) &= 0.022013 - 0.000754 \cdot I, \\ p''(s_0) &= -0.016527 + 0.004033 \cdot I. \end{aligned}$$

The coefficients f_1 and f_2 are computed as

$$f_1 = -0.220128 + 0.007537 \quad f_2 = 0.254099 - 0.815687 \cdot I.$$

This yields as a first order upper bound for proportional control

$$\bar{R} = 9.080,$$

while taking the improved upper bound yields

$$\bar{R} = 2.267.$$

6 Conclusion

Using complex analysis, general upper bounds for real proportional controllers have been derived. The new bound improves the known one by a factor of at least 9, and is in contrast not dependent on the number of roots. A systematic way to improve the new bound has been outlined. The general bound may not be improved by any constant factor smaller $1/28$, while the first of the improved bounds may not be lowered by a factor smaller than $1/14$.

Using the methods introduced here and in [2] we will show how to derive perturbation bounds for fixed rational controller structures in subsequent work [1].

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