ON COEFFICIENT DIAMETERS OF REAL SCHUR-STABLE INTERVAL POLYNOMIALS

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Abstract: A new necessary, sharp condition for Schur-stability of real interval polynomials is established here. Moreover, a way to study perturbation effects on the coefficient range is outlined. The results may be used in a preprocessing rejection scheme when testing for robust Schur-stability.

Keywords: Schur-stability, real polynomials, structured perturbation, conformal mapping

1. INTRODUCTION

Given a real, rational transfer function $P(z) = \frac{p_1(z)}{p_2(z)}$ it is of practical importance to constrain the positions of zeros and poles to a domain of stability. If the function considered is subject to perturbation, we face the following question: For which perturbations $\epsilon(z)$ of $p_i(z)$ will the zeros of $p_i(z) + \epsilon(z)$ be confined in the preset region?

In this presentation, negative results for structured perturbations regarding Schur-stability are exhibited. Especially, the first general upper bound on the possible perturbation range of an arbitrary single coefficient is proven. The bound just depends on the leading coefficient. It is shown to be sharp.

After briefly reviewing known results on real robust Schur-stability, the image of a rational function derived from the robustly stable polynomial is considered. The resulting image is mapped to the unit disc, and known coefficient bounds are applied to the resulting function.

2. KNOWN STABILITY CRITERIA

One obvious necessary condition for Schur-stability is that the gravity center of the roots must be inside the unit disc. Little seems to has been published on necessary conditions for robust Schur-stability, with the notable exception of (Blondel 1995).

A sufficient condition for all roots to remain inside the unit disc is necessarily dependent on all coefficients. This may be inferred from Viète's formula for the polynomial's coefficients. Numerous conditions may be found for example in (Marden 1966) or (Henrici 1974). For robust stability, well-known sufficient conditions have been given by Bose, Jury and Zeheb (Bose *et al.* 1986) and Soh, Berger and Dabke (Soh *et al.* 1985). While the approach of the former gives small, pairwise coupled, real intervals (which leave associated quadratic forms positive), the latter apply optimization principles to exhibit a hypersphere in complex coefficient space. The exact (symmetric) perturbation bound for real robust, Schur-stable polynomials has been expressed by Hinrichsen and Pritchard (cf. (Hinrichsen and Pritchard 1992), Cor. 8.2 or (Hinrichsen and Pritchard 1989), Cor. 4.4) as follows:

Given a real stable polynomial $p(z) = \sum_{j=0}^{n} a_j z^j \in \mathbf{R}[z]$ and a (real) coefficient perturbation-structure $c(z) = \sum_{j=0}^{n} c_j z^j \in \mathbf{R}[z]$. Define for $z \in \mathbf{C}$

$$G(z) := \frac{c(z)}{p(z)} =: G_R(z) + i \cdot G_I(z),$$

where $G_R(z), G_I(z) \in \mathbf{R}(z)$. The maximal allowable bound $r_{\mathbf{R}}$ for a single *real* parameter ρ such that $p(z)+\rho \cdot c(z)$ remains Schur-stable may then be computed as

$$r_{\mathbf{R}} = r_{\mathbf{R}}(p,c) = \min\{\frac{1}{|G(z)|} : |z| = 1, G_I(z) = 0\}.$$

It seems hard to estimate thereof $r_{\mathbf{R}}$ if only the perturbation of a single coefficient is known and no information on the other coefficients is given. Yet carrying out our above outlined program, we find that the diameter of any coefficient (i.e. the maximal allowable perturbation) is bounded by a small universal constant.

3. MEROMORPHIC FUNCTIONS FROM SCHUR-STABLE POLYNOMIALS

Starting with a stable family it is possible to construct a function which avoids a continuum of values. This may be seen to be in perspective with approaches of Ghosh, Blondel and others connecting simultaneous stabilization of systems and avoidance of value sets, as, e.g., in (Ghosh 1988), (Blondel 1994).

Consider an interval family of real polynomials, i.e. the collection of $p(z) = \sum_{j=0}^{n} a_j z^j$, where $a_j \in \left[a_j^-, a_j^+\right] \subset \mathbf{R}$. Assume $a_n^- > 0$ (so that the family is degree-invariant). Suppose every element of the family is Schur-stable. Thus, every family member $p(z)$ has all its roots inside the unit disc. Equivalently, every $p^*(z) := \sum_{j=0}^{n} a_{n-j} z^j = z^n p\left(\frac{1}{z}\right)$ has all zeros outside the closed unit disc $\overline{\mathbf{D}}$. Hence, Schur-stability of all family members is equivalent to the following condition for all $a_{n-j} \in \left[a_{n-j}^-, a_{n-j}^+\right]$,

$$\forall z \in \overline{\mathbf{D}} : p^*(z) = \sum_{j=0}^{n} a_{n-j} z^j \neq 0. \qquad (1)$$

Consider now the special family member

$$p^-(z) := a_n^- + \sum_{j=1}^{n} a_{n-j}^- z^j.$$

Define, for all k,

$$d_k := a_{n-k}^+ - a_{n-k}^- \; (\geq 0).$$

Hence, for $z \neq 0, d_k > 0$ the rational function $p^-(z)/(d_k \cdot z^k)$ takes no values in $[-1, 0]$, as otherwise $p^-(z) + \tau \cdot d_k \cdot z^k = 0$ for some $\tau \in [0,1]$, contradicting (1).

Switching the role of nominator and denominator yields

$$f_k(z) := \frac{d_k \cdot z^k}{p^-(z)} \notin (-\infty, -1] \quad \forall z : |z| < 1. \; (2)$$

The above function is holomorphic inside the unit circle as the denominator is free from zeros according to (1). The image of $f_k(z)$ is contained in the slit plane $M := \mathbf{C} \setminus (-\infty, -1]$.

Consider now the mapping of the maximal image domain M to the unit circle. As $\frac{1+z}{1-z}$ maps the unit disc conformally onto a halfplane, the function $\tilde{\lambda}(z) := \frac{z}{(1-z)^2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right]$ maps the unit disc onto the slit complex plane $\mathbf{C} \setminus \left(-\infty, -\frac{1}{4}\right]$. Hence,

$$\lambda(z) := c_0 + 4\frac{(c_0+1) \cdot z}{(1-z)^2} \qquad (3)$$

maps the unit disc isomorphically onto the slit plane $M = \mathbf{C} \setminus (-\infty, -1]$ for any non-negative c_0. The mapping is thus invertible. Hence, composition of λ^{-1} and f_k will yield a mapping of the unit circle.

Using the definitions (2), (3) given above (with $c_0 := f_k(0)$), define

$$g(z) := \lambda^{-1}(f_k(z)). \qquad (4)$$

If $d_k > 0$, $f_k(z)$ omits $(-\infty, -1]$ according to (2), the image of f_k is hence contained in $\mathbf{C} \setminus (-\infty, -1]$. Taking now $c_0 := f_k(0)$, it is obvious from (2) that $c_0 \geq 0$, as $a_n^- > 0, d_k \geq 0$. As $\lambda(0) = c_0$, we have $\lambda^{-1}(c_0) = 0$. The image of f_k thus lies inside the domain of definition of λ^{-1}. Thus, the image of the unit circle by $g(\cdot) = \lambda^{-1}(f_k(\cdot))$ lies inside the unit circle. This yields altogether

$$|g(z)| < 1 \quad \forall z \in \mathbf{D}, \qquad (5)$$
$$g(0) = \lambda^{-1}(f_k(0)) = \lambda^{-1}(c_0) = 0. \qquad (6)$$

4. FUNCTIONS OF BOUNDED MODULUS

Consider now the expansion of $\lambda^{-1}(z)$ around zero, say $\lambda^{-1}(z) = \lambda_0 + \lambda_1 \cdot z + \lambda_2 z^2 + \cdots$. We calculate the coefficients by the implicit function theorem, first of all

$$\lambda_1 = \frac{1}{1!} D_z^{(0)} \left(\frac{z}{\lambda(z) - c_0}\right)^1_{z=0} = \frac{z}{\lambda(z) - c_0} |_{z=0}$$
$$= \frac{(1-z)^2}{4(c_0+1)}\bigg|_{z=0} = \frac{1}{4(c_0+1)}. \qquad (7)$$

Suppose now $k \geq 1$. Thus, $c_0 (= f_k(0)) = 0$. Let the power series expansion of the composite function be given by $g(z) = \lambda^{-1}(f_k(z)) = \sum_{v=0}^{\infty} g_v z^v$. From

$$\lambda^{-1}(f_k(z)) = \lambda_1 \frac{z^k \cdot d_k}{p^-(z)} + \lambda_2 z^{2k} \left(\frac{d_k}{p^-(z)}\right)^2 + \cdots$$
$$= \sum_{v=0}^{\infty} g_v z^v \tag{8}$$

it is obvious that $g_0 = 0 = g_i \ \forall i \leq k-1$. The first non-vanishing coefficient ($d_k > 0$) is

$$g_k = \frac{1}{k!} \left(\lambda^{-1}(f_k(z))\right)^{(k)}_{z=0}$$
$$= \lambda_1 \cdot \frac{d_k}{a_n^-} = \frac{1}{4(c_0+1)} \cdot \frac{d_k}{a_n^-} = \frac{d_k}{4 \cdot a_n^-}. \tag{9}$$

A well-known inequality of Cauchy's for the coefficients of functions of bounded modulus is the following (see for example (Henrici 1974)).

Lemma 1. Given a power series $g(z) = \sum_{v=0}^{\infty} g_v z^v$ analytic in the unit circle $\mathbf{D}$ such that $|g(z)| < 1$ $\forall z \in \mathbf{D}$. Then

$$\sqrt[v]{|g_v|} \leq 1 \ \forall v. \tag{10}$$

As $g(z)$ is bounded according to (5), the application of Cauchy's estimate to the coefficent computed in (9) yields the following.

Theorem 2. Given $n+1$ real intervals $[a_j^-, a_j^+]$, $j = 0, \ldots n$, where $a_n^- > 0$. Suppose every polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ with coefficients $a_j \in [a_j^-, a_j^+]$ is Schur-stable. Then the following holds true.

$$|a_l^+ - a_l^-| \leq 4 \cdot |a_n^-|, \ l = 0, \cdots, n-1.$$

Example 1: Consider the well-known example of Bose and Zeheb (cf. (Barmish 1994) or (Bose and Zeheb 1986)) for a varying gravity center $q \in [-17/8, 17/8]$:

$$p(z;q) = z^4 + q \cdot z^3 + \frac{3}{2} z^2 - \frac{1}{3}.$$

The cited authors compute the roots explicitly to show that $p(z, -\frac{17}{8})$ and $p(z, \frac{17}{8})$ are Schur-stable, while $p(z, 0)$ is not. The Theorem allows to proceed as follows: The considered interval is $[-17/8, 17/8]$, the diameter is 4.25, well exceeding 4. According to the Theorem the polynomial $p(z;q)$ is identified as unstable - without computing the roots.

Remark: i) Our result was inspired by the work of Blondel on robust stable polynomials omitting two values. In (Blondel 1995), a bound connecting the range of the *leading* coefficient to the absolute value of the second coefficient was established.

ii) In (Batra 2003), a preliminary version of the new bound in Theorem 2 was established for the second but leading coefficient. A generalization of the symmetrization/subordination results used there leads to a dimension-dependent factor (of maximal n) in the diameter bound (cf. (Batra 2001)). This factor has been avoided here using the presented new approach.

The diameter bound holds for all real Schur-stable interval polynomials of arbitrary degree. It may be asked how conservative this criterion is. In fact, it may be shown to be sharp. Precisely: The above uniform diameter bound is sharp at least for the second coefficient's diameter, $|a_{n-1}^+ - a_{n-1}^-|$, as the following example shows.

Example 2: Consider the family of polynomials of degree $n \geq 2$, given as the real polynomials $k(z) = (1+\epsilon) \cdot z^n + q \cdot z^{n-1} + z^{n-2}$, where q varies in the interval $[-2, 2]$ and ϵ is positive. The family is Schur-stable, as the non-zero roots belong to the quadratic $(1+\epsilon) \cdot z^2 + q \cdot z + 1$ and lie inside the unit circle. Normalizing the family to be monic, the second coefficient's diameter becomes $4/(1+\epsilon)$. Thus, the constant 4 in Theorem 2 is best possible (Communicated by S.M. Rump).

Remark: The coefficient bound (10) may be applied to higher coefficients of the expansion (8) as well. This will yield bounds depending on coefficients and diameters as information is available. Consider the following sample.

Example 1 (cont'd): With $c_0 = 0$, compute λ_2 via the implicit function theorem to find $\lambda_2 = -\frac{1}{8}$. The second coefficient of

$$g(z) := \lambda^{-1}(f_k(z)) = \sum_{v=0}^{\infty} g_v z^v$$

is thus generally given by

$$g_2 = -\frac{1}{4} \frac{a_{n-1} \cdot d}{a_n^2} - \frac{1}{8} \frac{d^2}{a_n^2} = -\frac{1}{4} \frac{d}{a_n} \left(\frac{a_{n-1}}{a_n} + \frac{1}{2} \frac{d}{a_n}\right).$$

The use of Cauchy's estimate $|g_2| \leq 1$ allows to obtain limits whenever either the diameter d of a_{n-1} or the absolute value $|a_{n-1}|$ are preassigned. Supposing, for example, the diameter to be the possible maximum, i.e. $d = 4 \cdot a_n = 4, (a_n = 1)$ this shows the allowable coefficient range to be at most: $-3 \leq a_{n-1} \leq -1$. The range for a_{n-1} limits in turn the range for a_{n-2}. This may be turned into an algorithmic adaptive scheme for preprocessing polynomial families.

5. CONCLUSION

Using complex analysis, a sharp, constant bound for the coefficient diameter of real Schur-stable interval polynomials has been established. The bound holds uniformly for all polynomials and all coefficients. The result is based on mapping and avoidance properties of rational functions related to the family. It may be used as a check criterion for stability given only limited information, and for the explicit study of perturbation effects on the allowable coefficient range.

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