NEW POINT ESTIMATES FOR NEWTON'S METHOD

PRASHANT BATRA *

Abstract. The Newton iteration is customarily used for (sequential) approximation of zeros of differentiable functions. Beside the classical Kantorovich theory there exist convergence criteria which only involve data at one point, i.e. point estimates. The sufficient conditions ensure immediate quadratic convergence to a single zero and have been frequently used by different authors to design robust, fast and efficient root-finding methods for polynomials.

In this paper a sufficient condition for the simultaneous convergence of the one-dimensional Newton iteration for polynomials will be given. The new condition involves only n point evaluations of the Newton correction and the minimum mutual distance of approximations to ensure 'simultaneous' quadratic convergence to the pairwise distinct n roots.

To establish the new convergence condition a new error estimate for Newton's method will be proven. This estimate is applicable if n (rough) approximations of the pairwise distinct roots are well separated. The resulting error estimate is independent of n.

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1. Introduction. The well known work of Ostrowski and Kantorovich yields convergence criteria for a twice differentiable function f, whenever $\left|\left|Df(z_0)^{-1} \cdot f(z_0)\right|\right|$ and $\left|\left|D^2 f(z) \cdot Df(z_0)^{-1}\right|\right|$ can be suitably estimated on a circle around z_0 . Starting afresh, Myong-Hi Kim and Steve Smale independently derived sufficient convergence conditions for the Newton method from data at one point, see [12] or [13]. Before we formulate these conditions for holomorphic functions, two definitions (of Smale's) are appropriate.

Definition 1.1. *Given a holomorphic function $f : \mathbf{C} \to \mathbf{C}$. If for all Newton iterates z_k of $z_0 \in \mathbf{C}$: $|z_{k+1} - z_k| \le (\frac{1}{2})^{2^k - 1} |z_1 - z_0|, k = 0, 1, \cdots$, then the element $z_0 \in \mathbf{C}$ is called an approximate zero $(of\ f)$.*

The convergence of Newton's method from an approximate zero is obviously quadratic. An existence criterion for approximate zeros can be established via a simultaneous estimation of the Newton corrrection and certain coefficients of a Taylor expansion. For this purpose define for a holomorphic function f the following.

Definition 1.2. $\alpha(z, f) := |f(z)/f'(z)| \sup_{k>1} |\frac{f^{(k)}(z)}{k! f'(z)}|^{1/(k-1)}$.

If $\alpha(z_0, f)$ is suitably small, the starting point z_0 is an approximate zero, as shown by Kim and Smale. The condition involves only evaluation of functions at a single point. Using the majorant sequence technique, Xing-Hua Wang and Dan-Fu Han established a sharp result [14] which yields the following theorem.

Theorem 1.3. *Given a holomorphic function f and $z_0 \in \mathbf{C}$, suppose*

$$\alpha(z_0, f) < 3 - 2\sqrt{2}. \tag{1.1}$$

Then z_0 is an approximate zero.

For a polynomial, this sufficient condition involves only computable data. Based on such results, new algorithms for polynomial root approximation have been devised, see for example [7], [8]. These methods apply the Newton iteration sequentially for each root. On the other hand, there exist several methods for the simultaneous

* Inst. f. Informatik III, Technical University Hamburg-Harburg, Eißendorfer Str. 38, 21071 Hamburg, Germany (batra@tu-harburg.de).

approximation of polynomial zeros which are based on variants of Newton's method in $\mathbf{C}^n$. In this paper, a practical condition will be established which guarantees the convergence of Newton's method from n pairwise different approximations to n pairwise different zeros. Before presenting the details we sketch the way to derive such conditions.

Given a polynomial $P(z) = a_n \cdot \prod(z - \zeta_i)$ of degree n, consider the formal identity (derived in Section 3 as (3.2))

$$(1.2)\quad \alpha(z,P) = |\frac{P(z)}{P'(z)}| \cdot \max_{k>1} |\Big(\sum_{\substack{i_1<i_2<\cdots<i_k \\ i_1,i_2,\cdots,i_k=1}}^{n} \frac{1}{(z-\zeta_{i_1})\cdots(z-\zeta_{i_k})} \Big) \frac{P(z)}{P'(z)}|^{1/(k-1)}.$$

If $z = z_0$ is close to ζ_1, and well separated from the other zeros $\zeta_2, \cdots, \zeta_n$, condition (1.1) holds true, i.e. the Newton iteration converges starting with $z = z_0$. How to guarantee the separation?

Given z_1 with $P(z_1) \cdot P'(z_1) \neq 0$, there is the well-known estimate for the distance to the closest root, ζ_1 say:

$$(1.3)\qquad \left|\frac{P'(z_1)}{P(z_1)}\right| = \left|\sum_{i=1}^{n} \frac{1}{z_1 - \zeta_i}\right| \leq \frac{n}{|z_1 - \zeta_1|} \quad \text{or} \quad |z_1 - \zeta_1| \leq n \cdot \left|\frac{P(z_1)}{P'(z_1)}\right|.$$

Combining Theorem 1.3, (1.2) and (1.3) it is possible to prove a sufficient convergence condition like

$$(1.4)\qquad \max_i \left|\frac{P(z_i)}{P'(z_i)}\right| < \frac{\min_{i\neq j}|z_i - z_j|}{5 \cdot n^2}$$

for the convergence from the n points z_i to the n pairwise distinct zeros of $P(z)$ [2].

A second line of investigation starts from good practical error estimates for simultaneous methods. For example, it has been shown that the Durand-Kerner iteration $z_{i+1} := z_i - \frac{P(z_i)}{a_n \cdot \prod_{j\neq i}(z_i - z_j)}$ is convergent if

$$(1.5)\qquad \max_i \left|\frac{P(z_i)}{a_n \cdot \prod_{j\neq i}(z_i - z_j)}\right| < \frac{\min_{i\neq j}|z_i - z_j|}{2 \cdot n}$$

holds true, see [3]. As the Durand-Kerner iteration is Newton's iteration for Viète's system of equations [5], it seems natural to look for weaker conditions than (1.4) in the one-dimensional case as well. The sufficiency of condition (1.5) for the Durand-Kerner method has been established using a quite sharp error estimate from [4]. A new error estimate will be used here to establish a new sufficient convergence for Newton's method. This condition improves on (1.4) by $O(n)$, from $5n^2$ to $8n$ in the denominator.

For the general case, the estimation (1.3) is sharp. Generally it cannot be used to determine the number of zeros in the disc $I_i := \{z \in \mathbf{C} : |z - z_i| \leq n \cdot \left|\frac{P(z_i)}{P'(z_i)}\right|\}$. But if the discs I_i are mutually disjoint, each disc contains exactly one zero, and one may ask whether there exist better estimates in this case. The calculation necessary to determine whether two discs I_i, I_j are disjoint can trivially be modified to yield a relative measure of separation. The measure of separation can be used to improve the estimation (1.3). This leads to new inclusion discs $\hat{I}_i \subset I_i$ for the roots. Recursive application of this idea will lead to the following result in Section 2.

THEOREM 1.4. *Given a polynomial $P(z)$ of degree $n \geq 3$. Given n values z_i with $P'(z_i) \neq 0$. If* $\max_i \left|\frac{P(z_i)}{P'(z_i)}\right| < \frac{\min_{i\neq j}|z_i - z_j|}{C\cdot n}$, *$C > 2$, then each circle* $|z - z_i| < \frac{C}{C-1}\cdot\left|\frac{P(z_i)}{P'(z_i)}\right|$ *contains a zero.*
This estimate will be derived by completely elementary means. The new error estimate is used in Section 3 to prove the following new convergence condition which merely involves data from Newton's method.

THEOREM 1.5. *Given a polynomial $P(z)$ of degree $n \geq 3$. Given n values z_i with $P'(z_i) \neq 0$. Assume that*

$$\max_i \left|\frac{P(z_i)}{P'(z_i)}\right| < \frac{\min_{i\neq j}|z_i - z_j|}{8\cdot n}. \tag{1.6}$$

Then the following holds true.

i) The Newton iteration converges for each z_i, and convergence is quadratic.

ii) There is exactly one zero in $K_i := \{z \in \mathbf{C} : |z - (z_i - \frac{P(z_i)}{P'(z_i)})| \leq |\frac{P(z_i)}{P'(z_i)}|\}$.

Remark. Given n approximations z_i, to check the general convergence criterion (1.1) the n^2 values $\frac{P^{(k)}(z_i)}{k!P'(z_i)}$ for $k = 0, 2, \cdots, n;\ i = 1, \cdots, n$ have to be calculated. The new criterion only involves the evaluation of n Newton corrections $\frac{P(z_i)}{P'(z_i)}$ and improves on (1.4) by $O(n)$.

A theorem similar to Theorem 1.5 could be derived by assuming that a certain isolation of a single zero is already guaranteed. Estimations of that kind have, for example, been applied in the analysis of Pan's algorithm for the improvement of Weyl's quadtree construction as stated in [11].

2. A special error estimate. We restate the very well known estimation (1.3) from the introduction:

LEMMA 2.1. *Given a polynomial $P(z)$ of degree n. Given $z_0 \in \mathbf{C}$ with $P'(z_0) \neq 0$. The circle $|z - z_0| \leq n \cdot |\frac{P(z_0)}{P'(z_0)}|$ contains at least one root of $P(z)$.*

Without further assumption the inclusion radius $n \cdot |\frac{P(z_0)}{P'(z_0)}|$ of Lemma 2.1 is optimal, but not in the situation of Theorem 1.5. If the set of roots can be separated into at least two clusters and if the relative size of separation is known, this information can be used directly in the proof of Lemma 2.1. This observation leads to the following theorem.

THEOREM 2.2. *Given $P(z) = a_n \cdot \prod_{i=1}^n (z - \zeta_i) \in \mathbf{C}[z]$, $a_n \neq 0$, and z_1 with $P'(z_1) \neq 0$. Assume that the roots $\zeta_1, \cdots, \zeta_k$ ($k \leq n$) lie in* $D_1 := \{z \in \mathbf{C} : |z - z_1| \leq n \cdot |\frac{P(z_1)}{P'(z_1)}|\}$, *and all other roots in the exterior of D_1, i.e.,*

1.) $\zeta_i \in D_1$ for $1 \leq i \leq k$,

2.) $\exists S > 1\ :\ |z_1 - \zeta_i| \geq S \cdot n \cdot |\frac{P(z_1)}{P'(z_1)}|$ for $k + 1 \leq i \leq n$.

Then the circle

$$|z - z_1| \leq \frac{k}{n - (n-k)/S} \cdot n \cdot |\frac{P(z_1)}{P'(z_1)}|$$

contains a root.

Proof. The cases $n = 1$ or $P(z_1) = 0$ are trivial. Assume $n > 1$ and $P(z_1) \neq 0$. The roots are assumed to be numbered such that ζ_1 is the root with minimal distance to z_1, i.e. $|z_1 - \zeta_1| = \min_i |z_1 - \zeta_i|$. Define $\delta_m := \frac{Snk+(n-k)\delta_{m-1}}{Sn} = k + \frac{n-k}{Sn}\delta_{m-1}$ and

$\delta_0 := n$. First, it will be shown by induction that the circle

$$|z - z_1| \le \delta_m \cdot |\frac{P(z_1)}{P'(z_1)}| \tag{2.1}$$

contains a root for every m. The claimed inclusion (2.1) is guaranteed for $m = 0$ by Lemma 2.1. Assume that (2.1) holds true for m, i.e. $|z_1 - \zeta_1| \le \delta_m \cdot |\frac{P(z_1)}{P'(z_1)}|$. Remember that the roots are numbered such, that for all i with $2 \le i \le k$ it holds $|z_1 - \zeta_i| \ge |z_1 - \zeta_1|$. By assumption 2.) it is $|z_1 - \zeta_i| \ge Sn \cdot |\frac{P(z_1)}{P'(z_1)}|$ for $k+1 \le i \le n$, and therefore $|z_1 - \zeta_i| \ge \frac{Sn}{\delta_m} \cdot |z_1 - \zeta_1|$.
These estimates together show

$$\begin{aligned} |\frac{P'(z_1)}{P(z_1)}| &= |\sum_{i=1}^{n} \frac{1}{z_1 - \zeta_i}| \\ &\le \sum_{i=1}^{k} |\frac{Sn}{Sn(z_1 - \zeta_1)}| + \sum_{i=k+1}^{n} |\frac{\delta_m}{Sn(z_1 - \zeta_1)}| = \frac{Snk + (n-k)\delta_m}{Sn} \frac{1}{|z_1 - \zeta_1|}. \end{aligned}$$

Therefore, $|z_1 - \zeta_1| \le \frac{Snk+(n-k)\delta_m}{Sn} \cdot |\frac{P(z_1)}{P'(z_1)}| = \delta_{m+1} \cdot |\frac{P(z_1)}{P'(z_1)}|$. The inclusion (2.1) is established by induction. As $\frac{n-k}{Sn} < 1$, the sequence $\{\delta_m\}$ converges to the fixpoint $\hat{\delta} := \frac{Snk}{Sn-(n-k)} > 0$. □

To obtain estimates suitable for the named applications, Theorem 2.2 will be applied for all zeros simultaneously. The improved inclusion implicitly causes a stronger separation, and this leads to recursive improvement.

Proposition 2.3. *Given $P(z) = a_n \cdot \prod_{i=1}^{n}(z - \zeta_i) \in \mathbf{C}[z]$ of degree $n \ge 3$, and n values z_i with $P'(z_i) \ne 0$. Assume that for some $C > 2$,*

$$\max_i |\frac{P(z_i)}{P'(z_i)}| < \frac{\min\limits_{i \ne j} |z_i - z_j|}{C \cdot n} \ .$$

Let $E_0 := 1$ and $\rho_0 := C - 1$, and define two sequences by

$$E_{\nu+1} := \frac{1}{n - \frac{n-1}{\rho_\nu E_\nu}}, \qquad \rho_{\nu+1} := \frac{\rho_0 + 1}{E_{\nu+1}} - 1. \tag{2.2}$$

With suitable enumeration of the roots, the following holds true for all ν:

i) $\zeta_i \in \{z \in \mathbf{C} : |z - z_i| \le E_\nu \cdot n \cdot |\frac{P(z_i)}{P'(z_i)}|\}$ for all i.

ii) $\zeta_j \in \{z \in \mathbf{C} : |z - z_i| \ge \rho_\nu \cdot E_\nu \cdot n \cdot |\frac{P(z_i)}{P'(z_i)}|\}$ for all $j \ne i$.

The proof by induction is based on the following result.

Lemma 2.4. *Given $P(z) = a_n \cdot \prod\limits_{i=1}^{n}(z - \zeta_i) \in \mathbf{C}[z]$ of degree $n \ge 3$. Given n values z_i with $P'(z_i) \ne 0$. Assume there exist $E > 0$, $\rho \ge 1$ with $\rho \cdot E > 1$, such that*

1.) $\zeta_i \in \{z \in \mathbf{C} : |z - z_i| \le E \cdot n \cdot |\frac{P(z_i)}{P'(z_i)}|\}$ for each i,

2.) $\zeta_j \in \{z \in \mathbf{C} : |z - z_i| \ge \rho \cdot E \cdot n \cdot |\frac{P(z_i)}{P'(z_i)}|\}$ for each $j \ne i$.

Define $\hat{\delta} := \frac{\rho \cdot E \cdot n}{\rho \cdot E \cdot n - (n-1)}$. Then the following inclusion holds true:

$\zeta_i \in \{z \in \mathbf{C} : |z - z_i| \le \hat{\delta} \cdot |\frac{P(z_i)}{P'(z_i)}|\}$ for all i.

Proof. For any index i, the assumptions of Theorem 2.2 are satisfied for $k = 1$: 1.) by Lemma 2.1, 2.) with $S := \rho E$. An induction argument yields a sequence of radii with fixpoint $\hat{\delta}$ as in the proof of Theorem 2.2. □

Proof. [of Proposition 2.3] By assumption, $n \geq 3$, $\rho > 1$ and $\rho_0 \cdot E_0 > 1$. Expanding the defining formulas, it is obvious that the sequence $\{E_\nu\}$ is monotonically decreasing and the sequence $\{\rho_\nu\}$ is monotonically increasing. Moreover, $\rho_\nu \cdot E_\nu > 1$. By Lemma 2.1 at least one root is contained in each circle $|z - z_i| \leq n \cdot |\frac{P(z_i)}{P'(z_i)}|$. These circles are mutually disjoint, and each contains exactly one zero, which is obvious from

$$\begin{aligned}\min_{u \neq v} |z_u - z_v| &> (C-1) \cdot n \cdot |\frac{P(z_i)}{P'(z_i)}| + n \cdot |\frac{P(z_j)}{P'(z_j)}| \\ &> n \cdot |\frac{P(z_i)}{P'(z_i)}| + n \cdot |\frac{P(z_j)}{P'(z_j)}| \,, \end{aligned} \tag{2.3}$$

for arbitrary i, j. Hence, for each approximation z_i the assumptions of Lemma 2.4 are satisfied with $E := E_0$, $\rho := \rho_0$, therefore $i), ii)$ hold true for $\nu = 0$. Assume, that the claimed estimates hold for $\nu = \nu_0$. Then, with a fixed enumeration, the assumptions of Lemma 2.4 are satisfied with $E := E_\nu$, $\rho := \rho_\nu > 1$ because of $\rho_\nu \cdot E_\nu > 1$. This leads to the inclusion estimate

$$\zeta_i \in \{z \in \mathbf{C} : |z - z_i| \leq E_{\nu+1} \cdot n \cdot |\frac{P(z_i)}{P'(z_i)}|\}. \tag{2.4}$$

The exclusion region in $ii)$ is a trivial consequence. □

The limits of the sequences $\{E_\nu\}, \{\rho_\nu\}$ in Proposition 2.3 yield new error estimates as well. Theorem 1.4 will be a simple consequence of the following result.

Theorem 2.5. *Given $P(z) = a_n \cdot \prod_{i=1}^{n}(z - \zeta_i) \in \mathbf{C}[z]$ of degree $n \geq 3$, and n values z_i with $P'(z_i) \neq 0$. Assume that for some $C > 2$,*

$$\max_i |\frac{P(z_i)}{P'(z_i)}| < \frac{\min_{i \neq j} |z_i - z_j|}{C \cdot n}.$$

With $\rho_0 := C - 1$ and $\hat{E} = \left(\frac{1}{n} + \frac{n-1}{n} \cdot \frac{2}{\rho_0 \cdot n + 2 \cdot \sqrt{(\frac{\rho_0 \cdot n}{2})^2 - (n-1)}}\right)$ the following holds true (for a suitable enumeration of the z_i):

i) The root ζ_i lies in the circle

$$|z - z_i| \leq \hat{E} \cdot n \cdot |\frac{P(z_i)}{P'(z_i)}|,$$

ii) and for $j \neq i$, the root ζ_j lies in

$$|z - z_i| \geq \left[C - \hat{E}\right] \cdot n \cdot |\frac{P(z_i)}{P'(z_i)}|,$$

where

iii) $\hat{E} < \frac{C}{C-1}$.

Proof. Define the two sequences $\{\rho_\nu\}, \{E_\nu\}$ as in (2.2). We find, that $\rho_\nu \cdot E_\nu > 1$, and that both sequences are monotonic. By Proposition 2.3 we have the following properties of the sequences. With suitable fixed enumeration it holds for each index $i = 1, \cdots, n$ and every iteration index $\nu \geq 0$:

1.) Each root ζ_i is isolated in the circle $|z - z_i| \leq E_\nu \cdot n \cdot |\frac{P(z_i)}{P'(z_i)}|$.

2.) The roots ζ_j, $j \neq i$, lie in the region $|z - z_i| \geq \rho_\nu \cdot E_\nu \cdot n \cdot |\frac{P(z_i)}{P'(z_i)}|$.

If $P(z_i) = 0$ for some i, then $i), ii)$ hold true for that index, according to the assumption and Lemma 2.1. Otherwise, we obtain a strictly monotonic sequence of inclusion radii $\hat{E} \cdot n \cdot |\frac{P(z_i)}{P'(z_i)}|$ with a non-zero limit. Therefore, the sequences E_ν, ρ_ν are bounded. Passing to the limits $\hat{E}$ and $\hat{\rho}$ retains the inclusion-exclusion properties from $1.), 2.)$. It remains to calculate the limits. The recursion formula gives

$$\hat{E} = \frac{\hat{\rho} \cdot \hat{E}}{\hat{\rho} \cdot \hat{E} \cdot n - (n-1)}, \tag{2.5}$$

$$\hat{\rho} = \frac{\rho_0 + (1 - \hat{E})}{\hat{E}} = \frac{C - \hat{E}}{\hat{E}}. \tag{2.6}$$

Rearrangement of (2.5) shows

$$\hat{E} = \frac{\hat{\rho} + (n-1)}{\hat{\rho} \cdot n}. \tag{2.7}$$

Substitution in (2.6) shows

$$\begin{aligned} \hat{\rho} &= \frac{\rho_0 + (1 - \frac{\hat{\rho} + (n-1)}{\hat{\rho} \cdot n})}{\frac{\hat{\rho} + (n-1)}{\hat{\rho} \cdot n}} \\ &= \frac{\hat{\rho} \cdot \rho_0 \cdot n + \hat{\rho} \cdot n - \hat{\rho} - (n-1)}{\hat{\rho} + (n-1)}. \end{aligned}$$

This gives the quadratic equation

$$\hat{\rho}^2 - \hat{\rho} \cdot \rho_0 \cdot n = -(n-1).$$

Therefore, $\hat{\rho}$ is either of $\frac{\rho_0 \cdot n}{2} \pm \sqrt{(\frac{\rho_0 \cdot n}{2})^2 - (n-1)}$. Moreover,

$$\frac{\rho_0 \cdot n}{2} - \sqrt{(\frac{\rho_0 \cdot n}{2})^2 - (n-1)} < \rho_0.$$

As the sequence $\{\rho_\nu\}$ is monotonically increasing, its limit $\hat{\rho}$ is given by

$$\hat{\rho} = \frac{\rho_0 \cdot n}{2} + \sqrt{(\frac{\rho_0 \cdot n}{2})^2 - (n-1)}\,.$$

Substituting this in (2.7) shows

$$\hat{E} = \frac{1}{n} + \frac{n-1}{n} \cdot \frac{1}{\hat{\rho}} = \frac{1}{n} + \frac{n-1}{n} \cdot \frac{2}{\rho_0 \cdot n + 2 \cdot \sqrt{(\frac{\rho_0 \cdot n}{2})^2 - (n-1)}}. \tag{2.8}$$

Passing in 1.), 2.) from E_ν, ρ_ν to $\hat{E}, \hat{\rho}$ respectively yields $i), ii)$. The inequality $(\frac{1}{n} + \frac{n-1}{n} \cdot \frac{2}{\rho_0 \cdot n + 2 \cdot \sqrt{(\frac{\rho_0 \cdot n}{2})^2 - (n-1)}}) \cdot n < \frac{C}{C-1}$ is easily verified for $\rho_0 = C - 1$, which yields $iii)$. $\square$

3. Simultaneous convergence of Newton's method. Given a polynomial P of degree n, the function $\alpha(z,f)_{|f=P}$ reads

$$\alpha(z,P) = |\frac{P(z)}{P'(z)}| \max_{k=2,\cdots,n} |\frac{P^{(k)}(z)}{k!P'(z)}|^{1/(k-1)}.$$

Recall that by Theorem 1.3 the Newton iteration converges quadratically starting from z_0, if $\alpha(z_0,P) < 3-2\sqrt{2}$.

The sufficient convergence condition $\alpha(z_0,P) < 3-2\sqrt{2}$ involves for a polynomial of degree n only n point evaluations of rational functions, i.e., the condition only relies on 'attainable data'. But the essential values $\frac{P^{(k)}(z)}{k!P'(z)}$ which have to be calculated, are not necessary in the actual computation. A new sufficient convergence condition for Newton's method will be established which only involves information 'naturally' available from the Newton iteration.

As before, denote the zeros of $P(z)$ by $\zeta_1,\cdots,\zeta_n$. To employ the new estimate from Theorem 1.4, a different expression for $\alpha(z,P)$ is useful. Suppose $P(z)\neq 0$. Then

$$\begin{aligned}
&|\frac{P^{(k)}(z)}{k!P'(z)}| \\
&\quad = |\frac{1}{k!}P^{(k)}(z)\cdot P(z)^{-1}\cdot\frac{P(z)}{P'(z)}| \\
&\quad = |\frac{1}{k!}\Big(\sum_{i_1=1}^{n}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{n}\cdots\sum_{\substack{i_k=1\\ \forall m<k: i_k\neq i_m}}^{n}\prod_{\substack{j=1\\ j\neq i_1,i_2,\cdots,i_k}}^{n}(z-\zeta_j)\Big)\Big(\prod_{j=1}^{n}(z-\zeta_j)\Big)^{-1}\frac{P(z)}{P'(z)}| \\
&\quad = |\frac{1}{k!}\cdot k!\Big(\sum_{\substack{i_1,i_2,\cdots,i_k=1\\ i_1<i_2<\cdots<i_k}}^{n}\prod_{\substack{j=1\\ j\neq i_1,i_2,\cdots,i_k}}^{n}(z-\zeta_j)\prod_{j=1}^{n}(z-\zeta_j)^{-1}\Big)\frac{P(z)}{P'(z)}| \\
&\quad = |\Big(\sum_{\substack{i_1,i_2,\cdots,i_k=1\\ i_1<i_2<\cdots<i_k}}^{n}\frac{1}{(z-\zeta_{i_1})(z-\zeta_{i_2})\cdots(z-\zeta_{i_k})}\Big)\frac{P(z)}{P'(z)}|. \qquad (3.1)
\end{aligned}$$

By (3.1), the function $\alpha(z,P)$ may be expressed formally ($P'(z)\cdot P(z)\neq 0$) by

$$\begin{aligned}
\alpha(z,P) &= |\frac{P(z)}{P'(z)}|\cdot\max_{k>1}|\frac{P^{(k)}(z)}{k!P'(z)}|^{1/(k-1)} \\
&= |\frac{P(z)}{P'(z)}|\cdot\max_{k>1}|\Big(\sum_{\substack{i_1<i_2<\cdots<i_k\\ i_1,i_2,\cdots,i_k=1}}^{n}\frac{1}{(z-\zeta_{i_1})\cdots(z-\zeta_{i_k})}\Big)\frac{P(z)}{P'(z)}|^{1/(k-1)}. \qquad (3.2)
\end{aligned}$$

If approximations z_i of the roots ζ_i are given, lower estimates of $|z_i-\zeta_j|, j=1,\cdots,n$ facilitate the estimation of $\alpha(z_i,P)$ from (3.2).

Consider a fixed index i. Choosing a suitable numeration of the approximations z_j, assume $|z_i-\zeta_i| = \min_j |z_j-\zeta_i|$. Then, a lower estimate of $|z_i-\zeta_i|$ in the non-trivial case $P'(z_i)\cdot P(z_i)\neq 0$ can be obtained from lower estimates of $|z_i-\zeta_j|, j\neq i$ and an upper estimate of $|z_i-\zeta_i|$ as follows. Using the relation

$$(z_i-\zeta_i)\cdot\frac{P'(z_i)}{P(z_i)} = \sum_{j=1}^{n}\frac{z_i-\zeta_i}{z_i-\zeta_j} \quad \text{or} \quad (z_i-\zeta_i) = \Big(1+\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{z_i-\zeta_i}{z_i-\zeta_j}\Big)\cdot\frac{P(z_i)}{P'(z_i)}$$

gives

$$|z_i - \zeta_i| \geq \Big(1 - \frac{(n-1)|z_i - \zeta_i|}{\min_{j\neq i}|z_i - \zeta_j|}\Big) \cdot |\frac{P(z_i)}{P'(z_i)}|, \tag{3.3}$$

which can be utilized for a lower estimate.

The well-known Lemma 2.1 shows $|z_i - \zeta_i| \leq n \cdot |\frac{P(z_i)}{P'(z_i)}|$. So it is quite obvious, that conditions like (1.4) can be used to estimate $\alpha(z, P)$ by estimating suitably the distances $|z_i - \zeta_j|$ in (3.2). The use of Theorem 1.4 improves the estimates involved, so we are already prepared to prove Theorem 1.5.

Proof. [of Theorem 1.5] Assume $P(z_i) \neq 0$. Let $N := \max_j \left|\frac{P(z_j)}{P'(z_j)}\right|$. By Theorem 1.4 with $n \geq 3$ (and $C = 8$), a fixed enumeration can be chosen such that the following estimates hold true.

$$|z_i - \zeta_i| < \frac{8}{7} \cdot |\frac{P(z_i)}{P'(z_i)}| \ \leq \ \frac{8}{7} \cdot N \quad \text{for each } i, \tag{3.4}$$

$$|z_i - \zeta_j| \geq |z_i - z_j| - |z_j - \zeta_j| > 8(n - \frac{1}{7})N \quad \text{for each } j \neq i. \tag{3.5}$$

These estimates together with (3.3) yield

$$|z_i - \zeta_i| > \Big(1 - \frac{(n-1)\frac{8}{7} \cdot N}{8(n-\frac{1}{7}) \cdot N}\Big) \cdot |\frac{P(z_i)}{P'(z_i)}| > \frac{6}{7} \cdot |\frac{P(z_i)}{P'(z_i)}|. \tag{3.6}$$

To prove convergence, $\alpha(z_i, P)$ will be estimated for each $i \in \{1, \cdots, n\}$. It is useful to split the sum in (3.2) dependent on i:

$$\begin{aligned} &\sum_{\substack{i_1<i_2<\cdots<i_k \\ i_1,i_2,\cdots,i_k=1}}^{n} \frac{1}{(z-\zeta_{i_1})(z-\zeta_{i_2})\cdots(z-\zeta_{i_k})} \\ = &\sum_{\substack{i_1<i_2<\cdots<i_k \\ i_1,i_2,\cdots,i_k=1 \\ i\in\{i_1,\cdots,i_k\}}}^{n} \frac{1}{(z-\zeta_{i_1})\cdots(z-\zeta_{i_k})} + \sum_{\substack{i_1<i_2<\cdots<i_k \\ i_1,i_2,\cdots,i_k=1 \\ i\notin\{i_1,\cdots,i_k\}}}^{n} \frac{1}{(z-\zeta_{i_1})\cdots(z-\zeta_{i_k})} \\ =: &\sum{}' + \sum{}''. \end{aligned} \tag{3.7}$$

Using this notation together with (3.2), the function $\alpha(z, P)$ can be written as

$$\alpha(z, P) = \left|\frac{P(z)}{P'(z)}\right| \max_{k>1} \left|(\sum{}' + \sum{}'')\frac{P(z)}{P'(z)}\right|^{1/(k-1)}. \tag{3.8}$$

For fixed i and $k \geq 2$, the sum $\sum'$ $(i \in \{i_1, \cdots, i_k\})$ comprises exactly $\binom{n-1}{k-1} \leq (n-1)^{k-1}$ terms, the complementary sum $\sum''$ comprises $\binom{n-1}{k} < (n-1)^k$ terms accordingly.

The convergence of Newton's method can be deduced after appropiate estimation of $\alpha(z_i, P)$. Let $\mu := \min_{j\neq i}|z_i - \zeta_j|$. From (3.7) with (3.4), (3.5) and (3.6) it follows

$$\Big|\sum{}' + \sum{}''\Big| < (n-1)^{k-1} \cdot \frac{1}{|z_i - \zeta_i|}(\frac{1}{\mu})^{k-1} + (n-1)^k \cdot (\frac{1}{\mu})^k$$

$$
\begin{aligned}
&< |\frac{P'(z_i)}{6/7 \cdot P(z_i)}|(\frac{(n-1)}{8(n-\frac{1}{7})N})^{k-1} + (\frac{n-1}{8(n-\frac{1}{7})N})^k \\
&< |\frac{P'(z_i)}{6/7 \cdot P(z_i)}|(\frac{1}{8 \cdot N})^{k-1} + (\frac{1}{8 \cdot N})^k ,
\end{aligned}
$$

$$
\begin{aligned}
\left|\left(\sum{}' + \sum{}''\right)\frac{P(z_i)}{P'(z_i)}\right| &< \frac{7}{6}(\frac{1}{8 \cdot N})^{k-1} + (\frac{1}{8 \cdot N})^k \cdot |\frac{P(z_i)}{P'(z_i)}| \\
&\le \frac{7}{6}(\frac{1}{8 \cdot N})^{k-1} + \frac{1}{8} \cdot (\frac{1}{8 \cdot N})^{k-1} \\
&= \Big(\frac{7}{6} + \frac{1}{8}\Big)(\frac{1}{8 \cdot N})^{k-1}. \qquad (3.9)
\end{aligned}
$$

Hence, $\alpha(z_i, P)$ may be estimated with (3.8), (3.9) as

$$
\begin{aligned}
\alpha(z_i, P) &< \Big|\frac{P(z_i)}{P'(z_i)}\Big| \cdot \max_{k>1} \Big|\Big(\frac{1}{8 \cdot N}\Big)^{k-1}\Big(\frac{7}{6} + \frac{1}{8}\Big)\Big|^{1/(k-1)} \\
&\le \frac{1}{8} \cdot \Big(\frac{7}{6} + \frac{1}{8}\Big). \qquad (3.10)
\end{aligned}
$$

This yields $\alpha(z_i, P) < 3-2\sqrt{2}$, and Theorem 1.3 implies that z_i is an approximate zero. By Definition 1.1, the quadratic convergence is guaranteed. As $\sum_{k=0}^{\infty}(1/2)^{2^k-1} = 5/3$, all Newton iterates of z_i lie in $\{z \in \mathbf{C} : |z - z_i| \le 5/3 \cdot |\frac{P(z_i)}{P'(z_i)}|\}$. By (3.4) there is a zero in $\{z \in \mathbf{C} : |z - z_i| \le 8/7 \cdot |\frac{P(z_i)}{P'(z_i)}|\}$. The inclusion region with center $z_i - \frac{P(z_i)}{P'(z_i)}$ stated in Theorem 1.5, *ii*) is a simple consequence. These circles are pairwise disjoint. ∎

Remark. A different proof shows that the new criterion (1.6) can be relaxed to the domain of linear convergence.

4. Conclusion. A sufficient criterion for quadratic convergence of Newton's method has been established using a new error estimate. This criterion involves only n Newton corrections, and it improves by $O(n)$ on sufficient conditions directly computed from the point estimate theory of Kim and Smale.

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