On Gol'dberg's constant A_2

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Abstract

Gol'dberg considered the class of functions analytic in the unit disc with unequal positive numbers of zeros and ones there. The maximum modulus of zero- and one-places in this class is non-trivially bounded from below by the universal constant A_2. This constant determines a fundamental limit of controller design in engineering, and has applications when estimating covering regions for composites of fixed point free functions with schlicht functions. The lower bound for A_2 is improved in this note by considering simultaneously the extremal functions f and $1-f$ together with their reciprocals.

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1 Introduction

Landau showed in [11] that there exists a universal constant $R = R(a_0, a_1)$ such that all functions $f(z) = a_0 + a_1 z + \ldots$ analytic in the disc of radius R must attain one of the values 0 and 1 in this disc. This result was generalised by Schottky who showed in [15] that for an analytic function $f(z)$ which omits 0 and 1 on $|z| = r < 1$ there exists a modulus limit depending merely on $f(0)$ and r.

It was Gol'dberg [6] who discussed the minimum maximum modulus problem of zero- and one-places under the general form considered in this note. Given a function f holomorphic inside the unit disc $\mathbb{D}$. Let $n_0 := |\{z \in \mathbb{D} : f(z) = 0\}|$, $n_1 := |\{z \in \mathbb{D} : f(z) = 1\}|$. Denote the class of holomorphic functions f on $\mathbb{D}$ such that $0 \neq n_0 \neq n_1 \neq 0$ by K_2. Denote by $r(f)$ the maximum modulus of the zero- and one-places of $f \in K_2$. Denote by A_2 the greatest lower bound for all $r(f)$ with $f \in K_2$. Gol'dberg [6] showed that the universal constant A_2 is strictly positive, and proved the estimate

$$A_2 > 0.0000038.$$

An example is used in [6] to establish an upper bound to A_2. Gol'dberg claims that the example provides the upper bound

$$0.031 > A_2.$$

Related results of Jenkins' [10] imply that $A_2 > 0.00037$, while Blondel, Rupp and Shapiro [3] established

$$r(f) > \exp\left(-(1 + \frac{2}{\pi \cdot e})\frac{\pi^2}{N}\right), \text{ where } N := N(f) := \min\{n_0, n_1\}. \quad (1)$$

The bound (1) improves on Jenkins' bound for given f whenever f is such that $N \geq 5$.

The author established recently [1] that

$$A_2 > 0.00075.$$

The quest for narrow bounds on the universal constant A_2 can be motivated by mathematical as well as engineering consequences. As Rupp [14] showed, the composite of a normalized schlicht function with a fixed point free function covers a disc dependent in radius on Gol'dberg's constant A_2. This quantity is also of interest in the engineering context of stabilization.

The existence of certain stabilizing, time-invariant controllers (i.e. meromorphic functions) may be shown to be equivalent to the existence of a function with prescribed zeros and ones in a generalized circle. Several benchmark problems were posed by Blondel in [4] relating controller design to the value distribution of functions. For a reformulation of the control question in terms of the value distribution of functions, see [3].

The known lower bound to A_2 due to Gol'dberg as well as the one in [3] rely on quantitative maximum modulus estimates in Schottky's theorem in combination with the Poisson-Jensen formula. In this note, we deviate from this approach combining maximum modulus estimates with minimum modulus estimates on parametrized circles.

The outline of this note is as follows. After a short overview on quantitative estimates for Schottky's theorem, a mapping κ from $\mathbb{D}$ to the annulus $r < |z| < 1$ (where $r = r(f)$ is the maximum modulus of zero- and one-places of $f \in K_2$) is outlined. Using the composition of $f \in K_2$ with this mapping $|f(z)|$ and $|1-f(z)|$ are estimated on $|z| = \sqrt{r}$ taking the following steps. On every circle of radius ρ, where $r < \rho < \sqrt{r}$, there exists at least one point $\tilde{z}$ such that at least one of the functions $f, 1-f \in K_2$ takes a real value not exceeding $1/2$. The zeros of f give rise to a Blaschke product B unimodular on a circle of radius no less than $\sqrt{r}$. The Hadamard-Borel-Carathéodory inequalities then allow us to estimate from above the minimum modulus of f on $|z| = \sqrt{r}$ considering $1/(f \cdot B)$ at the point $\tilde{z}$.

This yields a function value smaller in modulus than $1/2$ on the outer radius $\sqrt{r}$. Zhang's sharp, quantitative version of Schottky's theorem gives an improved maximum modulus estimate for $f(\kappa(\cdot))$, and hence for f inside an annulus.

The maximum modulus bound can be used to estimate the minimum modulus from below via the Hadamard-Borel-Carathéodory inequalities for the product of f with the zero-cancelling Blaschke product B. The lower bound for $|f|$ and $|1-f|$ must be small as can be seen by an application of the argument principle. A proper choice of parameters allows us to conclude that $A_2 > 0.0012$.

2 Magnitude estimates via Schottky's theorem

Quantitative versions of Schottky's theorem have been sought for a long time. Schottky [15] established the following fundamental result.

Theorem 1 *Let $g(z) = a_0 + a_1 \cdot z + a_2 \cdot z^2 + \ldots$ be regular in $|z| < r$ and unequal to 0 or 1 in this circle. Then*

$$|\ln(g(z))| < \frac{2^{24}}{\sqrt{\alpha}} \left(\frac{r}{r - |z|} \right)^4,$$

where $\alpha := \min\left\{ |\ln a_0|, |\ln(1-a_0)|, |\ln \frac{a_0 - 1}{a_0}| \right\}$.

Hayman [7] showed that under the assumptions of the preceeding theorem

$$|g(z)| < \frac{1}{16} \left(\mu e^{\pi}\right)^{\frac{1+r}{1-r}} \text{ for all } z \text{ with } |z| < r, \quad \text{where } \mu := \max\{|a_0|, 1\},$$

and moreover, that π may not be replaced by a smaller constant. Ostrowski [12] had shown earlier that $(1+r)/(1-r)$ is the precise asymptotic order for $|a_0| \to \infty$. Several different ways to obtain estimates dependent on $|a_0|$ were pursued by Hayman, Jenkins, Lai, Hempel and Zhang (cf. [8] for references).

As we wish to apply the modulus estimate to f and $1-f$ simultaneously, we consider Zhang's version [16] of Schottky's theorem which takes $|a_0|$ into account even for small modulus (and slightly improves on [9]).

Theorem 2 *Let g be a holomorphic function without zeros and ones inside the unit disc. Then*

$$|g(z)| \leq K(a, |z|), \textit{for all } z \in \mathbb{D}, \tag{2}$$

where $K(a, |z|) := \dfrac{1}{16} \exp\left(\dfrac{\pi^2}{\ln(16/a + b)} \cdot \dfrac{(1+|z|)}{(1-|z|)}\right)$, $a := |g(0)|$, $b := e^{\pi} - 16$.
In case that $K = K(a, |z|) > 1$, the right-hand side of inequality (2) can be replaced by $K(a, |z|) - b/16$.

Suppose the maximum modulus of the zero- and one-places of f is r. To employ Schottky's theorem it is useful to map the unit disc to the annulus

$$\mathbb{A} := \{z \in \mathbb{C} : r < |z| < 1\}$$

where f and $1 - f$ are zero-free. Following Gol'dberg (see also [3]) we map $\mathbb{D}$ onto the annulus as follows. Map the unit disc via the logarithmic function $\eta(z) = i \ln i \frac{1+z}{1-z}$ to the infinite vertical strip S of width π centered at $-\pi/2$. The strip S is mapped onto the annulus $\mathbb{A}$ by the power function $\nu(\xi) = r^{-(\xi/\pi)}$. By concatenation obtain the surjective mapping

$$\kappa : \mathbb{D} \to \mathbb{A},\ \kappa(z) := \nu(\eta(z)).$$

The segment $-\pi/2 + \lambda i$ with $-\pi^2/|\ln(r)| \leq \lambda \leq \pi^2/|\ln(r)|$ is mapped by ν to the circle $|z| = \sqrt{r}$. The pre-image of the segment point $\xi_0 = -\pi/2 + \lambda i \in S$ under η has maximum modulus $\tanh \lambda/2$. Hence, there exists a preimage of the curve $|z| = \sqrt{r} \subset \mathbb{D}$ under the map κ bounded in modulus (see [6], p.205, or [3], p.190, eqs.(1)-(5)) by

$$\tanh(\pi^2/2|\ln(r)|). \tag{3}$$

Compose κ with a function f unequal to zero or one in the annulus $\mathbb{A}$ to obtain a mapping $g(\cdot) := f(\kappa(\cdot))$ satisfying the assumptions of Theorem 2. Modulus estimates for $g(0) = f(\kappa(0))$ as required for numerical computations involving Theorem 2 will be derived in the next section.

3 Limits to Minimum and Maximum Modulus

A function f in K_2 has different positive winding numbers around 0 and 1 on a circle $|z| = \rho, r(f) < \rho < 1$. Hence, with $r := r(f)$, the image of every circle $|z| = \rho, r < \rho < 1$ under f crosses the section between 0 and 1. This gives

$$\min_{|z|=\rho, r<\rho<1} \{|f(z)|, |1 - f(z)|\} < 1. \tag{4}$$

Moreover, on any circle $|z| = \rho, r(f) = r < \rho < 1$ we may choose (dependent on ρ) f_ρ as one of the functions f and $1 - f$ such that

$$\forall_{\rho \in (r,1)}\ \exists_{\tilde{z}=\rho e^{i\phi}}\ \exists_{f_\rho \in \{f, 1-f\}}\ \ 0 < f_\rho(\tilde{z}) \leq \frac{1}{2}. \tag{5}$$

Especially for $\rho = \sqrt{r}$ we obtain $\tilde{z}$ with $|\tilde{z}| = \rho$ and $|f_\rho(\tilde{z})| \leq 1/2$. This estimate is improved for $r = 0.0012$ using a vital source of Schottky's theorem: functions zero-free and bounded in a disc $|z| < \sqrt{r}$ can be estimated on an interior circle $|z| = \rho, \rho < \sqrt{r} =: r_{ext}$ using Carathéodory's improved formulation (first in [5]) of the Borel-Hadamard inequalities [13].

Lemma 1 *For a holomorphic function h unequal to zero in the disc $|z| < r_{ext}$, and bounded there in modulus by M_h we have:*

$$\textit{for all } z \textit{ with } |z| \le \rho < r_{ext} : \left|\frac{h(0)}{h(z)}\right| \le \left|\frac{M_h}{h(0)}\right|^{\frac{2\rho}{r_{ext}-\rho}} ; \tag{6}$$

$$\textit{for all } z \textit{ with } |z| \le \rho < r_{ext} : \left|\frac{h(z)}{h(0)}\right| \le \left|\frac{M_h}{h(0)}\right|^{\frac{2\rho}{r_{ext}+\rho}} . \tag{7}$$

Assume w.l.o.g. that for f we have $n_0(f) < n_1(f)$ (otherwise consider $1 - f$ instead). Let the zeros $z_1, z_2, \ldots, z_{n_0}$ of $f \in K_2$ lying in $|z| \le r(f)$ be given by

$$z_j = \gamma_j \cdot re^{i\vartheta_0}, 0 \le \gamma_j \le 1, \vartheta_j \in [0, 2\pi], r := r(f),$$

and consider the product of $f(z)$ with Blaschke factors $B_j(z)$ chosen as

$$B_j(z) := \frac{\sqrt{r} - \sqrt{r}\gamma_j e^{-i\vartheta_j} \cdot z}{z - \gamma_j re^{i\vartheta_j}}.$$

Each Blaschke factor is unimodular on $|z| = \sqrt{r}$, and of modulus at most

$$\frac{\sqrt{r} - \sqrt{r}\sqrt{cr}}{\sqrt{cr} - r} \text{ for } |z| = \sqrt{cr}, 0 < r < c < 1, \tag{8}$$

as a discussion of the parameter γ_j shows.
Multiply the function $f(z)$ by $B(z) := \prod_{j=1}^{n_0} B_j(z)$ to cancel all roots inside the unit disc. The analytic function $f(z)B(z)$ is zero-free, hence we may apply the upper estimate (7) to $h(z) := 1/(f(z)B(z))$ to estimate its minimum modulus. First, from (7) we obtain the inequality

$$\left|\frac{1}{f(z)B(z)}\right| \le \left|\frac{1}{f(0)B(0)}\right|^{\frac{r_{ext}-\rho}{r_{ext}+\rho}} \cdot \max_{|w|=r_{ext}} \left|\frac{1}{f(w)B(w)}\right|^{\frac{2\rho}{r_{ext}+\rho}} , \text{ where } |z| = \rho < r_{ext}.$$

Therefore, taking reciprocals,

$$|f(z)B(z)| \quad \ge \quad |f(0)B(0)|^{\frac{r_{ext}-\rho}{r_{ext}+\rho}} \cdot \min_{|w|=r_{ext}} |f(w)B(w)|^{\frac{2\rho}{r_{ext}+\rho}} , \text{where } |z| = \rho < r_{ext}.$$

Since $B(z)$ is unimodular on $|z| = r_{ext} = \sqrt{r}$ the following inequality holds for an arbitrary $\tilde{z}$ of modulus ρ:

$$\min_{|z|=r_{ext}} |f(z)| \le |f(\tilde{z})|^{\frac{r_{ext}+\rho}{2\rho}} \max_{|z|=\rho} |B(z)|^{\frac{r_{ext}+\rho}{2\rho}} \left|\frac{1}{f(0)B(0)}\right|^{\frac{r_{ext}-\rho}{2\rho}} .$$

This implies, using the estimate (8) for the n_0 factors of the Blaschke product $B(z)$, and the relation $|1/B(0)| \le (\sqrt{r})^{n_0}$, that

$$\min_{|z|=r_{ext}} |f(z)| \le |f(\tilde{z})|^{\frac{r_{ext}+\rho}{2\rho}} \left| \left(\frac{\sqrt{r} - \sqrt{r}\cdot\rho}{\rho - r} \right)^{n_0} \right|^{\frac{r_{ext}+\rho}{2\rho}} \left| \frac{(\sqrt{r})^{n_0}}{f(0)} \right|^{\frac{r_{ext}-\rho}{2\rho}}. \qquad (9)$$

Put $r := 0.0012 \sim 10^{-2.92082}$, hence $r_{ext} := \sqrt{r} \sim 10^{-1.46041}$. Choose an intermediate value as $\rho = 10^{-2.1601} \sim 0.0069167$. By (4), there exists $\tilde{z}$ on the circle $|z| = \rho$ such that $|f(\tilde{z})| \le 1$. We may estimate the minimum modulus of f on $|z| = \sqrt{r} = r_{ext}$ distinguishing the following three cases.

1.) Assume $|f(0)| > 0.96155$. Then, from (9) (with $|f(\tilde{z})| \le 1$) we get

$$\begin{aligned} \min_{|z|=r_{ext}} |f(z)| &\le \left| \left(\frac{\sqrt{r} - \sqrt{r}\cdot\rho}{\rho - r} \right)^{n_0} \right|^{\frac{r_{ext}+\rho}{2\rho}} \left| \frac{(\sqrt{r})^{n_0}}{f(0)} \right|^{\frac{r_{ext}-\rho}{2\rho}} \\ &\le (6.0176872)^{3.004152} \left(\frac{(10^{-1.46041})^1}{0.96155} \right)^{2.004151} \le 0.2810411, \end{aligned}$$

since the upper bound decreases with growing $n_0 \ge 1$.
2.) If $|f(0)| < 0.5075$, consider $1 - f$ with modulus at the origin at least 0.4925, and $n_1 \ge 2$ zeros. Using the inequality (9) with $1 - f$ in place of f (for some $\tilde{z}$ with $|1 - f(\tilde{z})| \le 1$), and n_1 in place of n_0, we obtain

$$\begin{aligned} \min_{|z|=r_{ext}} |1 - f(z)| &\le \left| \left(\frac{\sqrt{r} - \sqrt{r}\cdot\rho}{\rho - r} \right)^{n_1} \right|^{\frac{r_{ext}+\rho}{2\rho}} \left| \frac{(\sqrt{r})^{n_1}}{1 - f(0)} \right|^{\frac{r_{ext}-\rho}{2\rho}} \\ &\le (36.2125592)^{3.004152} \left(\frac{(10^{-1.46041})^2}{0.4925} \right)^{2.004151} \le 0.279094, \end{aligned}$$

since the upper bound decreases with growing $n_1 \ge 2$.
3.) Consider finally the case $0.5075 \le |f(0)| \le 0.96155$. The function

$$F(z) := f(z) \cdot (1 - f(z))$$

has at least three zeros, while $|F(0)| \ge |\ 0.96155 \ \cdot \ (\ 1 \ - \ 0.96155\ )\ |$. By (4), it exists a point $\tilde{z}$ on the circle of radius $\rho = 10^{-2.1601}$ such that $|F(\tilde{z})| \le 1/4$. Using this information in (9) with F in place of f (hence $n_0 = n_0(F) \ge\ 3$) we obtain

$$\begin{aligned} \min_{|z|=r_{ext}} |F(z)| &\le |1/4|^{\frac{r_{ext}+\rho}{2\rho}} \max_{|z|=\rho} \left| \left(\frac{\sqrt{r} - \sqrt{r}\cdot\rho}{\rho - r} \right)^{n_0+n_1} \right|^{\frac{r_{ext}+\rho}{2\rho}} \left| \frac{(\sqrt{r})^{n_0+n_1}}{F(0)} \right|^{\frac{r_{ext}-\rho}{2\rho}} \\ &\le \left(\frac{(6.0176872)^3}{4} \right)^{3.004152} \left(\frac{(10^{-1.46041})^3}{0.03697159} \right)^{2.004151} \le 0.2021, \end{aligned}$$

since the upper bound decreases with growing $n_0 + n_1 \ge 3$.

This estimate of $|F(z)| = |f(z)(1-f(z))|$ implies that at least one of the functions f and $1-f$ is at one point of the circle $|z| = r_{ext}$ no larger in modulus than 0.281042.

The three cases considered above allow us to conclude that for at least one $\tilde{f} \in \{f, 1-f\}$ we have

$$\min_{|z|=r_{ext}} |\tilde{f}(z)| \leq 0.281042 < 0.28105.$$

We may assume that $g(\cdot) = \tilde{f}(\kappa(\cdot))$ has modulus at most 0.28105 at the origin, or else consider a suitable rotation of $\kappa(\cdot)$. This leads to the maximum modulus estimate of $\tilde{f} \in \{f, 1-f\}$ on $r < |z| < \sqrt{r}$ via (2) and (3) (with $b := e^{\pi} - 16$)

$$\max_{r<|z|<\sqrt{r}} |\tilde{f}(z)| \leq \frac{1}{16} \exp\left(\frac{\pi^2}{\ln(\frac{16}{0.28105} + b)} \exp\left(\frac{\pi^2}{|\ln(r)|}\right)\right) - \frac{b}{16} \leq 1844.702 =: M. \quad (10)$$

With this bound for $|\tilde{f}(z)|$ over $r < |z| < \sqrt{r}$, i.e. a bound for one of the functions $|f|$ and $|1-f|$, we obtain a lower bound for the modulus of f on a new circle $|z| = \rho$ (where $\rho \in (r, \sqrt{r})$ will be specified later) via the following estimate: inequality (6) for $h(z) := f(z)B(z)$, $r_{ext} = \sqrt{r}$ yields for z with $|z| = \rho$ that

$$|f(z)| \geq \left(\max_{|z|=\rho} |B(z)|\right)^{-1} \cdot |f(0)B(0)|^{\frac{\sqrt{r}+\rho}{\sqrt{r}-\rho}} \cdot \left(\max_{|w|=r_{ext}} |B(w) \cdot f(w)|\right)^{\frac{-2\rho}{\sqrt{r}-\rho}}.$$

As $B(w)$ is unimodular for $|w| = r_{ext} = \sqrt{r}$, we use (10) to obtain

$$|f(z)| \geq \left(\max_{|z|=\rho} |B(z)|\right)^{-1} \cdot |f(0)B(0)|^{\frac{\sqrt{r}+\rho}{\sqrt{r}-\rho}} \cdot (M+1)^{\frac{-2\rho}{\sqrt{r}-\rho}}. \quad (11)$$

With $n_0 \geq 1$ and $|f(0)| \geq 0.8985$, the Blaschke factor estimate (8) in (11) yields for all z with $|z| = \rho$ the lower bound

$$|f(z)| \geq \left(\frac{\rho - r}{\sqrt{r} - \sqrt{r} \cdot \rho}\right)^{n_0} \cdot \left[0.8985 \cdot \left(\frac{1}{\sqrt{r}}\right)^{n_0}\right]^{\frac{\sqrt{r}+\rho}{\sqrt{r}-\rho}} \cdot (M+1)^{\frac{-2\rho}{\sqrt{r}-\rho}}.$$

Choose the intermediate radius as $\rho := 10^{-2.155}$ to obtain for $r = 0.0012$ on $|z| = \rho$ the estimate

$$\begin{aligned} |f(z)| &\geq \left(\frac{\rho - r}{\sqrt{r} - \sqrt{r} \cdot \rho}\right)^{n_0} \cdot \left|f(0)\left(\frac{1}{\sqrt{r}}\right)^{n_0}\right|^{\frac{\sqrt{r}+\rho}{\sqrt{r}-\rho}} \cdot (M+1)^{\frac{-2\rho}{\sqrt{r}-\rho}} \\ &\geq 0.168565 \cdot |0.8985/\sqrt{0.0012}|^{1.506351} (1846)^{-0.506351} \geq 0.50413. \end{aligned}$$

Similarly $|1-f(0)| \geq 0.1015$ and $n_1(f) = n_0(1-f) \geq 2$ yield for all z on $|z| = \rho$ the lower estimate

$$\begin{aligned}
|1-f(z)| &\geq \left(\frac{\rho - r}{\sqrt{r} - \sqrt{r}\cdot\rho}\right)^{n_1} \cdot \left|(1-f(0))\left(\frac{1}{\sqrt{r}}\right)^{n_1}\right|^{\frac{\sqrt{r}+\rho}{\sqrt{r}-\rho}} \cdot (M+1)^{\frac{-2\rho}{\sqrt{r}-\rho}} \\
&\geq \left(\frac{\rho - r}{\sqrt{r} - \sqrt{r}\cdot\rho}\right)^{2} \cdot \left|(1-f(0))\left(\frac{1}{\sqrt{r}}\right)^{2}\right|^{\frac{\sqrt{r}+\rho}{\sqrt{r}-\rho}} \cdot (M+1)^{\frac{-2\rho}{\sqrt{r}-\rho}} \\
&\geq 0.168565^2 \cdot |0.1015/0.0012|^{1.506351} \, (1846)^{-0.506351} \geq 0.504207.
\end{aligned}$$

This contradicts the fact that the functions f and $1-f$ simultaneously intersect the interval $(0,1)$ on $|z| = \rho$, i.e. contradicts equation (5). This establishes the following result.

Theorem 3

$$A_2 > 0.0012.$$

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