LESEPROBE
An Improved Computation Method of the Seakeeping Green Function

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1. Introduction

The history of the study of ship motions and hydrodynamic properties using three-dimensional panel methods is long. After the first fundamental study of Haskind (1946), some decades were spent by many researchers to complete the theoretical studies. The fundamental mathematical formulation seems to be completed by the derivation of the line-integral term by Yamasaki (1970) and Brard (1970).

Numerical pioneering works following this theory were presented by Chang (1977), Inglis and Price (1981) and Kobayashi (1981). However, the insufficient power of computers at that time prevented exact and accurate calculations. Treatment of the line-integral term and/or consideration of steady disturbance effects were insufficient in these calculations, although some of the results showed improved estimation of the seakeeping compared to strip theory. The calculations also exposed the numerical difficulty of evaluating the Green function used in the integral equation as a kernel. This difficulty arises from the mathematical property of the Green function itself and is a fundamental drawback of the 3-D Green function method with forward speed. The presence of various kinds of mathematical expressions of this Green function presented by many researchers indicate not only the numerical difficulty on fast and accurate evaluation, but also the mathematical efforts to overcome this problem.

Bessho (1977) derived a successful mathematical expression of the Green function, which was expressed by the single integral of the elementary functions in the complex domain. This single integral expression has a definite advantage against the others which are expressed by using the exponential integral function $E_1$ from the point of view of computational accuracy and speed. On the other hand, the numerical integration must be performed in the complicated complex domain.

Iwashita and Ohkusu (1989) developed a special algorithm named ‘Numerical Steepest Descent Method’ for evaluating this function and succeeded to achieve a fast and accurate computation of the Green function. The most noteworthy point is the robustness of the method even if both field point and source point approach the free surface. The numerical integration does not waste CPU time or computational accuracy, as the integral path is taken such that the integrand never oscillates on the it. The method has been applied to the 3-D boundary value problem for many practical ships, Iwashita and Ohkusu (1989, 1992), Iwashita et al. (1992, 1993, 1994), Ito and Iwashita (1996). Some previously unnoticed problems in the 3-D Green function method became clear and were solved that way.

Recently, many other researchers presented computation results supported by the development of powerful computers. The development of the hardware of the computer will make the 3-D computations, including not only the 3-D Green function method but also the Rankine panel method, more popular and easier even on personal computers in the near future.

Notwithstanding the development of the hardware, more sophisticated Green-function algorithms are still expected. This report documents such an improvement of the computation method of the Green function based on the numerical steepest descent method of Iwashita and Ohkusu (1989). The complex domain is restricted only on the $\theta$-plane without transforming to the half-infinite $M$-plane, and the steepest descent line is interpolated by the Riesenfeld function based on the $B$-spline. The former makes it easy to evaluate the integrand without taking care of the branch cuts, and latter makes the numerical search of the steepest descent line more flexible and the numerical integration faster. The computation results are compared with the old method and the usual method with $E_1$ function, and efficiency of the present method is discussed.
2. The Green function

The Green function $G(x, y, z; x', y', z')e^{-i\omega t}$ which expresses the velocity potential at $(x, y, z)$ due to a point source located at $(x', y', z')$, advancing with constant speed $U$ and pulsating strength with circular frequency $\omega$, is defined as a function subject to the Laplace equation in the fluid domain $(z < 0)$ and the linearized unsteady free-surface boundary condition together with the radiation condition at infinity:

$$\nabla^2 G(x, y, z; x', y', z') = -\delta(x - x')\delta(y - y')\delta(z - z')$$  \hspace{1cm} (1)

$$\left[(i\omega - U \frac{\partial}{\partial x})^2 + \mu (i\omega - U \frac{\partial}{\partial x}) + g \frac{\partial}{\partial z}\right] G(x, y, z; x', y', z') = 0 \quad \text{on} \quad z = 0$$  \hspace{1cm} (2)

$G$ must be also zero as $z \to -\infty$. $\delta$ is a Delta function and $\mu$ denotes Rayleigh’s artificial viscosity introduced to satisfy the radiation condition at infinity. The Green function satisfying those conditions can be expressed as

$$G(x, y, z; x', y', z') = \frac{1}{4\pi} \left( \frac{1}{r} - \frac{1}{r'} \right) - \frac{i}{2\pi} K_0 T(X, Y, Z)$$  \hspace{1cm} (3)

where $K_0 = g/U^2$, $X = K_0(x - x')$, $Y = K_0|y - y'|$, $Z = K_0(z + z')$ and

$$r, r' = \sqrt{(x - x')^2 + (y - y')^2 + (z \mp z')^2}$$

The r.h.s. first term shows the Rankine term and its mirror image, and the second term is the wave term which includes the local wave and progressive waves.

Bessho (1977) showed that the wave term of the Green function can be expressed by the single integral as follows:

$$T(X, Y, Z) = \int_{-\pi}^{\pi} \frac{d\theta}{\sqrt{1 + 4\tau \cos \theta}} \left[k_2 e^{k_2 \omega} - \text{sgn}(\cos \theta) k_1 e^{k_1 \omega}\right]$$  \hspace{1cm} (4)

where

$$\frac{k_1}{k_2} = \frac{1}{2 \cos^2 \theta} \left(1 + 2\tau \cos \theta \pm \sqrt{1 + 4\tau \cos \theta}\right), \quad \omega = Z + i(X \cos \theta + Y \sin \theta),$$

$$\varphi = \cos^{-1} \frac{X}{\sqrt{X^2 + Y^2}}, \quad \varepsilon = \sinh^{-1} \frac{|Z|}{\sqrt{X^2 + Y^2}}, \quad \alpha = \left\{ \begin{array}{ll} \cos^{-1} 1/4\tau & (4\tau > 1) \\ -i \cosh^{-1} 1/4\tau & (4\tau < 1) \end{array} \right.$$

The partial derivatives with respect to $x, y$ and $z$ are easily evaluated as follows:

$$\nabla T(X, Y, Z) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} K_0 \tilde{T}(X, Y, Z)$$  \hspace{1cm} (5)

where

$$\tilde{T}(X, Y, Z) = \int_{-\pi}^{\pi} \begin{bmatrix} i \cos \theta \\ i \sin \theta \\ 1 \end{bmatrix} \frac{d\theta}{\sqrt{1 + 4\tau \cos \theta}} \left[k_2 e^{k_2 \omega} - \text{sgn}(\cos \theta) k_1 e^{k_1 \omega}\right] + \tilde{T}_0(X, Y, Z)$$  \hspace{1cm} (6)

$$\tilde{T}_0(X, Y, Z) = -\frac{X\sqrt{X^2 + Y^2 + Z^2 + iX|Z|^2}}{Y^2 + Z^2} \begin{bmatrix} -Y + \frac{i(X^2 + Y^2)\sqrt{X^2 + Y^2 + Z^2}}{X} \\ i(X^2 + Y^2)\sqrt{X^2 + Y^2 + Z^2} \end{bmatrix}$$  \hspace{1cm} (7)
The numerical method to evaluate the Rankine term in (3) has been developed by some authors and there is no numerical difficulty, e.g. Hess and Smith (1964), Webster (1975), and Newman (1986). Then our task is to evaluate the wave term (4) and its partial derivatives (6). The wave term expressed by \( T \)-function includes the \( k_2 \) wave and the \( k_1 \) wave. The former is generally a long wave like ring wave deformed due to the forward speed and the latter is a short wave like Kelvin wave deformed due to the periodical pulse of the source strength. Those waves also include the local wave near the source point besides the propagating waves.

For the \( k_2 \) wave system, the numerical integration can be easily performed along the straight line

\[
\text{for } (\tau < 4) : \quad -\pi - i \cosh^{-1} \frac{1}{4\tau} \rightarrow -\pi - i \cosh^{-1} \frac{1}{4\tau} + i\delta \rightarrow -\pi + \varphi \rightarrow -\frac{\pi}{2} + \varphi - i\varepsilon
\]

\[
\text{for } (\tau > 4) : \quad -\pi + \cos^{-1} \frac{1}{4\tau} \rightarrow -\pi + \cos^{-1} \frac{1}{4\tau} + \delta \rightarrow -\frac{\pi}{2} + \varphi \rightarrow -\frac{\pi}{2} + \varphi - i\varepsilon
\]

since the integrand does not oscillate with high frequency. Only the singularity of the magnitude of \( O(1/\sqrt{\varepsilon}) \), \( \varepsilon \rightarrow 0 \) at \( \vartheta = -\pi + \alpha \) has to be taken care of. This singularity, however, is small enough to apply the usual numerical integration scheme if we perform the transformation, so called double-exponential integral transformation, near this point such as:

\[
\theta = \frac{(B - A) \tanh(1.5 \sinh t) + (B + A)}{2} \quad (-\infty < t < \infty), \quad A = -\pi + \alpha, \quad B = A - \delta
\]

where \( |\delta| \) is taken small enough. In practice, \( |\delta| = 0.1 \) proved to be adequate and the numerical integration is truncated at \( |t| = 3.5 \) to avoid numerical underfloats. For the singularity near \( \vartheta = -\pi + \alpha \), the calculation of \( k_1 \) term is treated similarly.

For the \( k_1 \) wave system, the calculation has another fundamental problem. The integrand oscillates with high frequency as \( \theta \) approaches \( \pm \pi/2 \) along the previous path. Furthermore, if both the field point and the source point are close to the free surface, i.e. \( Z \approx 0 \), the amplitude of the integrand converges slower toward zero when \( \theta \rightarrow \pm \pi/2 \). Hence we face the integration of the function with high-frequency oscillation and large amplitude along the path, and generally the application of the usual numerical integration methods breaks down for this case. Some special approach is required.

The method of Iwashita and Ohkusu (1989) is a method to integrate such functions along the steepest descent line searched numerically. The steepest descent line is defined as a line on which the imaginary part of the argument of the exponential function, \( k_1 \varphi \), keeps constant and the real part decreases most rapidly. Along this line, the integrand does not oscillate any more. The usual numerical integration scheme is then easily applied without wasting CPU time and with guaranteed accuracy. Originally, this steepest descent integration method was applied only for the region close to \( \theta = \pm \pi/2 \), transforming the complex \( \theta \)-plane to the complex \( M \)-plane where the integration range is mapped to the half-infinite region. This transformation makes the numerical search of the steepest descent line easy when \( \theta \) approaches \( \pm \pi/2 \). However, we must take care of the presence of the branch cuts and this sometimes complicates the numerical search of the steepest descent line. In this report, we perform the integration on the \( \theta \)-plane without any transformation to another complex plane. This should shorten the CPU time for searching the steepest descent line numerically.

3. Numerical method

3.1 Complex \( \theta \)-plane

We consider only the term relating to \( k_1 \) in (4) and (6), and define the function \( I(\gamma) \) in the form

\[
I(\gamma) \equiv \int_{\gamma}^{\text{sgn}s_{\frac{1}{2}}} \psi(\theta) e^{\phi(\theta)} d\theta
\]

(8)
where

\[
\psi(\theta) = \begin{bmatrix} 1 \\ ik_1 \cos \theta \\ ik_1 \sin \theta \\ k_1 \end{bmatrix} \frac{k_1}{\sqrt{1 + 4 \pi \cos \theta}}
\]

(9)

\[
\phi(\theta) \equiv k_1 \omega = k_1 [Z + i(X \cos \theta + Y \sin \theta)]
\]

(10)

\[
\text{sgn} \equiv \text{sgn}[\sin(\mathbb{R}\gamma)]
\]

(11)

The property of the integrand near \( \theta = \pm \pi/2 \) mainly depends on the exponential function in (8), and its argument function \( \phi(\theta) \) presents an information of this exponential function. Fig. 1 shows the typical contour lines of the real part and the imaginary part of the function \( \phi(\theta) \).

The density of the contour line becomes higher near \( \theta = \pm \pi/2 \) due to the large magnitude of \( k_1 \) function which has the infinite amplitude at \( \theta = \pm \pi/2 \), and the distortion of the contour line is complicated.

![Contour lines of \( \phi(\theta) \)](image)

The integration range of the integrations (4) and (6) is concretely from \(-\pi + \alpha \equiv (-3.14159, -0.69315)\) to \(-\pi/2 + \varphi - i\varepsilon \equiv (1.32582, -0.02425)\) in this case. The function \( \phi \) always becomes zero at \( \theta = -\pi/2 + \varphi - i\varepsilon \) because \( \omega = 0 \) at this point. The path from \(-\pi + \alpha \) to \(-\pi/2 + \varphi - i\varepsilon \) can be taken freely because of the Cauchy’s theorem for the analytical functions provided the path is not taken to the direction where the function becomes infinite in amplitude. We can therefore select the steepest descent line near \( \theta = \pm \pi/2 \) instead of the path along the real axis or straight lines parallel to the imaginary axis. Fig. 1 shows one possible example of the integral path partly including the steepest descent lines. As shown in the figure, the steepest descent line is orthogonal with the contour line of the real part and parallel to that of the imaginary part. This means the line descends most rapidly without changing imaginary value of \( \phi \). The integrand therefore decreases exponentially without any oscillation relating to the exponential function along this line.

Here our task is at first to search the adequate path including the steepest descent line partly. For this purpose we must analyze the saddle points and the cross points between the real axis and the steepest ascent line which is through the saddle point. The black circles in Fig. 1 show the saddle points and the white circles on the real axis are the cross points with the steepest ascent line. There is the possibility for the steepest descent line to be taken to the wrong direction due
to the existence of the saddle points, because they can sometimes have two paths to descend for the same destination point, \( \theta = \pi/2 \) or \( \theta = -\pi/2 \). The correct descending direction can be analyzed mathematically and the details will be discussed later. The information about the saddle points and the cross points between real axis and the steepest ascent line are used to avoid such wrong paths.

### 3.2 Saddle Points

The saddle points are defined as points where the derivative of \( \phi(\theta) \) with respect to \( \theta \) is zero. Then we may obtain the saddle points by solving the following equation:

\[
\phi'(\theta) = \left[k_1(\theta)(Z + i(X \cos \theta + Y \sin \theta))\right]' = 0
\]  

(12)

The relation concerning the derivative of \( k_1(\theta) \)

\[
k_1'(\theta) = \frac{\sin \theta}{\cos \theta} \left(1 + \frac{1}{\sqrt{1 + 4\tau \cos \theta}}\right) k_1(\theta)
\]

(13)

can be used in (12), and we get easily

\[
(Z \sin \theta + iY) \sqrt{1 + 4\tau \cos \theta} = -\sin \theta[Z + i(X \cos \theta + Y \sin \theta)]
\]

(14)

Taking the square of both sides we have the equation to be solved as

\[
(Z \sin \theta + iY)^2(1 + 4\tau \cos \theta) + \sin^2 \theta[Z + i(X \cos \theta + Y \sin \theta)]^2 = 0
\]

(15)

If we put \( e^{i\theta} = t \), \( \sin \theta \) and \( \cos \theta \) becomes

\[
\cos \theta = \frac{t^2 + 1}{2t}, \quad \sin \theta = \frac{t^2 - 1}{2it}, \quad \text{where} \quad e^{i\theta} = t
\]

(16)

Substituting (16) into (15) leads to the following polynomial equation of complex domain

\[
(Y + iX)^2 t^8 - 4Z[2\tau Z - (Y + iX)] t^7 - 4Y[Y + iX - 8\tau Z] t^6
+ 4[2\tau(Z^2 - 4Y^2) + Z(Y - iX)] t^5 + 2(X^2 - 5Y^2) t^4 + 4[2\tau(Z^2 - 4Y^2) - Z(Y + iX)] t^3
- 4Y[Y - iX + 8\tau Z] t^2 - 4Z[2\tau Z + (Y - iX)] t + (Y - iX)^2 = 0
\]

(17)

Eight solutions obtained by solving (17) must be checked by using (14) after evaluating \( \cos \theta \) and \( \sin \theta \) using (16). Only four solutions can be expected for the adequate solution of (14). If \( t \) is determined as an adequate solution, \( \theta \) can be evaluated by

\[
\theta \equiv a + ib, \quad \text{where} \quad a = \eta, \quad b = -\log \xi, \quad t \equiv \xi e^{i\eta}
\]

(18)

Next we calculate the cross point between the real axis and the steepest ascent line which is through each saddle point obtained in (18). This gives us the information for the judgment of the genuine steepest descent line. We assume the imaginary part of \( \phi \) takes the value represented by \( f \), on the saddle point obtained by (18). The steepest ascent line through the saddle point means the line on which the imaginary part of \( \phi \) takes the same value as the saddle point and the real part ascends most rapidly. If this line crosses the real axis, the following equation must be satisfied on the real axis:

\[
\Im[\phi(\theta)] = k_1(X \cos \theta + Y \sin \theta) = f, \quad \text{where} \quad \theta \quad \text{and} \quad f \quad \text{are real}
\]

(19)

Putting \( t = e^{i\theta} \) and using (16), we have

\[
[\tau(X - iY) - f]^2 t^4 - 2f(X - iY) t^3 + 2[\tau(X - iy) - f] [\tau(X + iy) - f] t^2 - 2f(X + iy) t + [\tau(X + iy) - f]^2 = 0
\]

(20)