Basic Theory of Wave Analysis
(State of the Art 1975)

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ABSTRACT

The paper provides an evaluative survey on mathematical models which are underlying the current methods for determining wave resistance through wave pattern measurements. Emphasis is on elucidating implicit assumptions rather than analytical derivations. Special weight is given to such problems which are still open for future research. Some comments are made to questions related to the decay of the 'local' wave field.

NOMENCLATURE

- \( A, B, C, D \) = control surface for energy flux
- \( a \) = wave amplitude
- \( b \) = tank width
- \( c \) = waterline along a ship
- \( C(w), S(w) \) = Fourier transforms for longitudinal cut method
- \( C_2(w), S_2(w) \) = Fourier transforms for longitudinal cut method
- \( F(u), G(u) \) = sine, cosine component of free wave spectrum
- \( f(\theta), g(\theta) \) = alternative definition of free wave spectrum
- \( G(x, y, z; x'; y'; z') \) = Green's Function of point source of unit strength
- \( g \) = acceleration due to gravity
- \( H(k, \theta) \) = complex Kochin function
- \( H \) = complex conjugate of \( H \)
- \( n \) = water depth
- \( J(u, w) \) = alternative form of Kochin function
- \( J' \) = complex conjugate of \( J \)
- \( k \) = dimensionless circular wave number in direction \( \theta \)
- \( k_0 \) = fundamental wave number
- \( l = \omega \cos \theta \) = cosine component of wave number \( \omega \)
- \( m = \omega \sin \theta \) = sine component of wave number \( \omega \)
- \( t \) = unit vector normal to the ship
- \( p(w), q(w) \) = Fourier transform of \( p(x) \)
- \( p(x), q(x) \) = two measurable quantities
- \( R_w \) = wave resistance
- \( R_u \) = polar coordinates with respect to origin
- \( r \) = distance of flow point from source point
- \( r_1 \) = distance of flow point from image point of source
- \( S \) = control surface
- \( s = s(u) \) = function of \( u \)
- \( t \) = time
- \( U \) = ship speed
- \( u, v, w \) = flow component in horizontal, lateral and vertical direction
- \( u \) = \( k \sin \theta \) = dimensionless circular wave number induced in y-direction
- \( u_{w} \) = discrete value of \( u \) in tank of finite width
- \( w = k \cos \theta \) = dimensionless circular wave number induced in x-direction
- \( X \) = longitudinal component of horizontal wave force
- \( Y \) = transverse component of horizontal wave force
- \( x \) = coordinate in direction of motion of ship
- \( y \) = horizontal coordinate positive to portside
- \( z \) = vertical coordinate positive upwards
- \( (x, y, z) \) = flow point
- \( \alpha \) = Kelvin angle
- \( \varphi \) = wave elevation
- \( \Phi, \Phi_r \) = partial derivatives of \( \tau \)
- \( \theta \) = direction of wave propagation
- \( \omega = k \cdot \omega \) = wave number
- \( \rho \) = fluid density
- \( C \) = wave frequency
- \( \psi \) = perturbation potential
- \( q_x, q_y, q_z \) = components of perturbation velocity
As long as man has used naval transportation, he must have observed that moving floating bodies make waves, which reflect in some way part of the effort needed to keep a vessel advancing in the desired direction. But it was not earlier than almost 25 years ago that interest awakened to determine the power supplied to the wave pattern from the ship quantitatively through evaluating the wave pattern geometry.

Exploratory enterprise came up in this country [1] and in France [2] around 1950, but it was the truly pioneering work of Prof. Inui which cleared the soil for the second, more rational phase of research. He presented his achievements at the - I may now say historical - 18th meeting of the "H-5 panel" - that is the ship-wave committee of the American Society of Naval Architects - and this worked like an ignition spark for world wide simultaneous research.

The timing for the first international seminar on wave resistance theory in 1963 was just appropriate to have the attendance confronted with about ten different methods of wave analysis already. This set signal for the next phase, which meant synoptical analysis and generalisation together with systematic assessment of practical experience. Thus the variety of different approaches could be tied to and re-derived from common mathematical models.

It would be premature if I claim that this phase is facing its completion already, but I feel that meanwhile so much progress is evident that we are under an almost compulsory need to have the state of the art presented here and to be discussed cooperatively by potential experts. This is in particular so with respect to rapidly increasing demands facing the naval architect for most accurate prediction of powering requirement of ships.

My role to give an overall report on wave analysis theory within this seminar is made easier in so far as four well-arranged surveys on current theoretical development are already available: One [3] by Engers, Sharma and Ward (1967, referred to as ESW in the sequel), one (1969) by Ikenaha [4], one (1973) by Wehausen [5] and finally one (1975) by Good [6] within the report of the ICTC Resistance Committee. I take the liberty to assume that you had opportunity to study at least one of above papers so that I can concentrate on certain particular aspects which - I feel - have not been treated with adequate weight so far. Apart from reporting what has been achieved, I intend to compile material which still needs to be evaluated and completed, in order to find cross-correlations and to show gaps where further research is needed. For this purpose, I have built up a bibliographical appendix. Its aim is to update the literature given in ESW to cover all papers somehow related to problems of wave analysis since 1967. References to this appendix are made in standard form, i.e. by author and year of publication as is used in the bibliographical work of Wehausen and in ESW.

One basic assumption of all wave analysis methods is that viscosity is neglected in the following. However, there must be also kinetic energy. And the wave resistance, in which we are finally interested, results from the lengthwise component of total energy transport. The group velocity, which governs this process, depends on wave length of individual components. It cannot be expected as suggested by Korvin-Kroukovsky [12] that there is a direct relation between potential energy of some area under observation and the resistance of the ship creating it.

We can consider some models for the flow manifested through the wave pattern and derive wave resistance as function of the flow components. Inserting then relations between flow and wave elevation, we are aiming to express wave resistance through characteristics of the wave geometry only. This is actually possible for several models of wave flow, which in turn give reasonable approximations to the actual flow related to the wave pattern, at least away from certain domains close to the ship, where "local waves" can not be disregarded. This restriction comes up because these flow models can be accepted only under special assumptions. We mentioned already that there should be no vorticity, hence, a velocity potential $\psi$ can be introduced such that flow components $u,v,w$ may be expressed as its partial derivatives. It is further
necessary to require the smallness of wave elevation $\xi$ and flow components $Q_x, Q_y, Q_z$ in such a sense that a linearised free surface condition can be assumed to hold on the undisturbed free surface.

1.1 Expression for the Total Wave Pattern (Including Near-Field Effects)

Let us take a standard coordinate system to describe waves and the flow. The plane $z=0$ is the undisturbed free surface, the positive $x$ axis is in the direction where the ship advances steadily with speed $U$, and $z$ is positive upwards. We shall not omit ourselves to take a specific choice of origin regarding $y$, unless a symmetry plane $y=0$ can be found. The position of the coordinate origin regarding $x$ is arbitrary in so far as it will not influence in general the quality of approximations used; there is a special situation, however, in connection with Kelvin wave patterns 1.3.

Any arbitrary wave pattern $z = \xi(x,y)$ with sufficient decay at infinity will admit then a global double Fourier integral representation regarding $x$ and $y$. Anticipating later insight, we shall separate some particular denominator from the Fourier spectral function and set

$$\psi(k_u) = \frac{1}{4\pi^2 k_u^2} \text{Im} \left[ \int \int \frac{2\xi(x,y) c_i(x,y)}{w - i\epsilon x - i\epsilon y} \, dw \, dv \right]$$

(1a)

with $k_u = g/u^2$.

Thus the wave field is governed by some "Kochin function" $J(u,w)$ which is in general complex. It is convenient - or more instructive - to express $\xi$ through polar coordinates in the $x$-$y$ and $u$-$w$ plane equivalently as

$$\psi(k_u) = \frac{1}{4\pi^2 k_u^2} \text{Im} \left[ \int \int \frac{2\xi(e,\omega)}{w - i\epsilon x - i\epsilon y} \, dw \, dv \right]$$

(1b)

(1a) and (1b) are then interlinked through the system of relations,

$$x = R \cos \omega \quad w = k \cos \theta \quad \text{dudw} = k \, dx \, dy$$
$$y = R \sin \omega \quad u = k \sin \theta \quad \text{dudw}$$

(1c)

is in close conformity with the notation introduced by Kavelock [1] and by Kochin [14]. We shall write down our results in both notations simultaneously if considered helpful for better understanding.

The evaluation of (1a) near the line $w = u(u) \approx \left[(1 + \frac{1}{2} + \frac{1}{2})/2\right]^2$ (or the evaluation of (1b) near the line $k = \cos^2 \theta$) must be performed in terms of complex integration in such a manner that no far-field waves appear for $x$ positive. No special considerations will be required if $J(u,s(u))$ resp. $i(\cos^{-1} \theta, \theta)$ were identically zero. In this case $\psi(k_u)$ degenerates to a "wave free" wave pattern. In fact, it is only the

1.2 Wave Patterns represented through Single-Integral Expressions or through a Series

For regions sufficiently far behind the ship (i.e. for $x > x_s(t)$) we may deduce from (1) up to terms of order $O(x_b^{-1})$

$$\psi(k_u) = \frac{1}{4\pi^2 k_u^2} \text{Im} \left[ \int \frac{F(u) \sin k_u(x - u) + G(u) \cos k_u(x - u)}{\epsilon(x + u)} \, du \right]$$

with $G(u) + iF(u) = \frac{4\pi^2}{k_u^2} u^{-1} - J(u + i\omega)$

(2a)

In polar notation this leads to

$$\psi(k_u) = \frac{1}{4\pi^2 k_u^2} \text{Im} \left[ \int \frac{(g(u) \sin k_u(\cos^2 \theta \cos(\theta - \omega)) + g(\omega) \cos(\cos^2 \theta \cos(\theta - \omega))) \, d\theta}{\epsilon(\cos^2 \theta)} \right]$$

with $g(\theta) = i/\cos^2 \theta \cos^2 \theta$.

(2b)

(2b) makes clear that this is a representation of $\xi$ through a continuous system of plane waves ("free waves") with wave number $x$ ranging from $x_0$ to infinity and angle of propagation $\theta$ against $x$ axis between $\pi/2$ and $\pi/2$.

If the ship is moving in a canal with vertical walls along $y = \pm b/2$, the expression analogous to (2a) - as well as to (1a) - can be derived by evaluating the $u$-integration in the sense of a trapezoidal rule with a step width $\Delta u = 2\pi/k_u$ and $\theta$ ranging from $-\pi/2$ to $\pi/2$. We then obtain (up to $O(x_b^{-1})$)

$$\psi(k_u, \omega) = \frac{1}{4\pi^2 k_u^2} \text{Im} \left[ \int \frac{F(u) \sin k_u(\cos^2 \theta \cos(\theta - \omega)) + G(u) \cos(\cos^2 \theta \cos(\theta - \omega))) \, d\theta}{\epsilon(\cos^2 \theta)} \right]$$

with $u = u_0 + \Delta u$

(3a)

Inserting $G(u_0) + iF(u_0) = \frac{4\pi^2}{k_u^2} u_0^{-1} - J(u_0 + i\omega)$; $u_0 = \sin \theta_0 \cos \theta_0$

we have the polar representation

$$\psi(k_u, \omega) = \frac{1}{4\pi^2 k_u^2} \text{Im} \left[ \int \frac{F(u_0) \sin k_u(\cos^2 \theta \cos(\theta - \omega)) + G(u_0) \cos(\cos^2 \theta \cos(\theta - \omega))) \, d\theta}{\epsilon(\cos^2 \theta)} \right]$$

(3b)

We obtain (2a), (2b) from (3a), (3b) reversely with $b$ tending to infinity, hence $\Delta u \to du$. 
(2a) is the basis for the so called "transverse cut" wave analysis methods. (3a) is the starting point for the matrix method of Hoener (1972) and for the "multiple longitudinal cut" method of Moran and Landweber (1972). We have for simplicity assumed here that the wave pattern is symmetric to the plane y = 0, accordingly we must take F(ωu, Δu) = F(-ωu, Δu), G(ωu, Δu) = G(-ωu, Δu). The velocity potential associated to (2a) is

\[ \phi(x,y,z) = \frac{U}{\gamma_{\Lambda}} \int \left[ \frac{\rho(x) \cos k_p x + \mu(x) \sin k_p x}{\Delta^2} \right] \frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} \right)^n \left( \frac{\partial}{\partial y} \right)^m \frac{\gamma_{\Lambda}}{\gamma_{\Lambda}} \right] \]

from which the expression for finite tank width can be derived following above rules.

Considering now unrestricted water again, but taking \( y_0 \) sufficiently large, we may approximate \( \phi \sim \phi_{\Lambda} + \phi_{\Pi} \) up to order \( O(y_0^{-1}) \) by

\[ \phi_{\Lambda}(x,y,z) = \frac{1}{\gamma_{\Lambda}} \int \frac{1}{\omega^2} \left[ C_\omega \cos k_p x + \frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} \right)^n \left( \frac{\partial}{\partial y} \right)^m \frac{\gamma_{\Lambda}}{\gamma_{\Lambda}} \right] \right] \]

with \( C_\omega, S_\omega = e^{2\pi i k_n x} \omega (\omega^2 x^2 - x^2 - 1) \)

\[ \phi_{\Pi}(x,y,z) = -\frac{1}{\gamma_{\Lambda}} \Im \left[ \int \frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} \right)^n \left( \frac{\partial}{\partial y} \right)^m \frac{\gamma_{\Lambda}}{\gamma_{\Lambda}} \right] \right] \]

We shall give a corresponding polar representation only for \( \phi_{\Pi} \), as it displays an analogy to (2b), though with different upper limit of integration,

\[ \phi_{\Pi}(\rho,\theta) = -\frac{1}{\gamma_{\Lambda}} \int \left[ C_\omega \cos k_p \rho \cos \theta + \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \rho} \right)^n \left( \frac{\partial}{\partial \rho} \right)^m \frac{\gamma_{\Lambda}}{\gamma_{\Lambda}} \right] \right] \]

It was outlined in ESW that for each method of wave analysis just one particular wave flow model is pertinent. The approximate basis for "longitudinal cut" wave analysis is (5a). It is certainly true that (2a) could serve as an approximation as well for \( x \) sufficiently large, and Havlock's "variable integration limit" model (ESW p. 143) is valid in a formal sense. However, so long as the same restrictions as \( \mu(y) \), but only \( \phi(x,y) \) leads to a consistent result, perhaps due to being "uniformly valid" in a certain sense along a cut \( y \neq y_0 \). It needs some pondering to understand the joint action of the two components \( \phi_{\Lambda} \) and \( \phi_{\Pi} \). Whereas \( \phi_{\Pi} \) again is a system of free waves, the components of \( \phi_{\Pi} \) display non-oscillatory decay with increasing \( y_\Lambda \).

With (1) given, \( C_\omega, S_\omega, \gamma_{\Lambda} \) depend clearly on the choice of \( y \)-coordinate origin, as does the value of \( y_0 \). But for \( y = 0 \), \( \phi_{\Pi} \) - though convergent - can not be considered part of some "local disturbance" in so far as it contributes to far-field waves with \( x \) tending both to plus and minus infinity; so does \( \phi_{\Pi} \) in this case. An asymptotic analysis shows that for \( x \) positive, \( \phi_{\Pi} \) finally cancels \( \phi_{\Pi} \) - the range of value \( C_\omega(x) \) and \( S_\omega(x) \) for \( x \) less than 1 is irrelevant - whereas for \( x \) tending to minus infinity, \( \phi_{\Pi} \) duplicates \( \phi_{\Pi} \). If, however, \( y_0 \) is nonzero, no far-field contributions from \( \phi_{\Pi} \) can be found with \( |x| \) approaching infinity (Stoke's phenomenon of mathematical physics).

1.3 Slowly Varying Wave Trains, Kelvin Patterns in particular

If to (2) - or to (5) in case \( y \neq 0 \) - a second asymptotic evaluation is performed, a representation of \( \phi \) is found with no waves outside some wedge-shaped region \( x < 0, |y/x| \leq 1/\sqrt{8} \), whereas for each point in this region two systems of wave ("transverse" and "divergent") are present, whose characteristics \( \phi \) and \( \theta \) depend on space coordinates \( x \) and \( y \), or \( \rho \) and \( \theta \) if we use polar notation. Explicitly we obtain

\[ \phi(x,y) = \frac{1}{\gamma_{\Lambda}} \sqrt{\pi x \gamma_{\Lambda} + \gamma_{\Lambda}^2 x^2} \sum_{n=1}^{\infty} \cos \mu_{n+1} \cos \mu_{n-1} \cos (\phi_{\Lambda}(\rho,\theta) e^{2\pi i n \theta}\\(n=0) \cos (\phi_{\Lambda}(\rho,\theta) e^{2\pi i n \theta} - \frac{\pi}{2}) - 0 \]

This is the condition of stationary phase. In case of deep water, we have

\[ \phi_0(\rho) = \psi_\Lambda \cos \theta \]

so that \( \phi_1, \phi_2 \) are roots of \( \cos \theta = 2 \tan \theta + \cos \theta = 0 \), i.e.

\[ \tan \theta_{\phi_1} = -\frac{1}{2} \left[ \cos \theta - \sqrt{\cos^2 \theta - 8} \right] \]

and thereby \( \tan \theta_{\phi_1} = 1/\sqrt{2} \leq \tan \theta_{\phi_2} \). In case that (3) does not hold, e.g. for finite depth \( h \), factors of (6a), (6b) will be different.

It should be observed that in case \( \phi \) depends on \( \theta \) solely - in particular under (8) - for each of above wave systems the locus of constant \( \theta \) (and thus constant \( \phi \)), the so called "characteristic curves" comes out as straight lines, radiating from the origin within the wedge domain. A system of curved "wave crests" may be constructed by eliminating \( \theta \) from two equations (7) and

\[ \phi(x,y) \cos (\rho - \theta) = n \pi \]

as shown by Hogner [15] in the special