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Scattering
of charged particles by random electromagnetic fields

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Summary: A general, relativistic formalism for particle scattering by random electromagnetic fields is developed in the weak-interaction approximation. The scattering is described by a Fokker-Planck equation, the coefficients of which are derived as linear spectral integrals over an infinite set of resonance surfaces. The formulae are evaluated explicitly for three field models. The theory is applied to the spatial diffusion parallel to the mean field, the equilibrium pitch-angle distribution, and Fermi acceleration processes. Detailed comparisons with observations will be given in subsequent papers.


1. Introduction

Interactions between charged particles and fluctuating electromagnetic fields play an important role in a number of geo- and astrophysical problems. The particle distributions of the solar wind and the magnetosphere represent collisionless plasmas, for which the details of distributions are determined primarily by wave-particle rather than particle-particle interactions. Various scattering processes of high-energy particles in interplanetary space, as inferred from solar-flare particle propagation or the solar modulation of the galactic radiation, are also believed to have their origin in random irregularities in the interplanetary fields.

It appears worthwhile, therefore, to investigate in a general manner the influence of an arbitrary random electromagnetic field on the particle distributions of a magnetized plasma. Various aspects of the problem have been treated in recent papers by Jokipii (1966), Roelof (1966), Kennel and Petschek (1966), and others. In particular, it has been recognized that the evolution of the particle distributions can be described by a

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Fokker-Planck equation, the coefficients of which depend linearly on the field covariance functions or spectra. However, the functional dependence on the spectra has not been derived in closed form, so that estimates of the coefficients have usually been based on various rather questionable approximations. By using spectral representations throughout and applying the asymptotic resonance relations of wave scattering theory (c.f. Hasselmann, 1967), we shall show in the following that the Fokker-Planck coefficients can be derived in closed form as spectral integrals over an infinite series of equidistant resonance surfaces.

Resonant interactions occur between individual particles and Fourier components of the electromagnetic field whenever the “parallel” frequency (i.e., the Doppler shifted and Lorentz dilated frequency of the field component relative to the parallel particle motion) is equal to a multiple of the particle’s Larmor frequency. Special cases include the Landau resonance and the MHD- and Whistler-mode resonances considered by Dragt (1961), Wenzel (1961), Kennel and Petschek (1966) and, in a somewhat different context, by Jokipii (1966) and Roelof (1966). The wave number in these cases is parallel to the mean field. If the wave number is non-parallel, an infinite number of resonances occur. They are essentially the same resonances that occur in the theory of plasma waves (c.f. Stix, 1962), except that we are concerned here with their effect on the particle distributions rather than on the wave motion. Normally, the Fokker-Planck coefficients include comparable contributions from many resonances.

The theory is based on the weak-interaction approximation, which requires that \( \partial H / \partial \nu \ll HT \), where \( H \) is the electromagnetic spectrum, \( \nu \) is the frequency, and \( T \) is a characteristic transfer time. The condition does not depend explicitly on the Larmor frequency; it involves only the “smoothness” of the spectrum. The analysis is carried through explicitly as a perturbation about the guiding centre description, which requires additionally that the transfer time is large compared with the gyration period. In many applications, both conditions are satisfied. However, the guiding-centre approximation is not a basic limitation of the theory. In the case of a non-magnetized plasma, or a plasma with a weak mean field, an alternative Fokker-Planck equation can be derived by perturbing about the zero’th order distributions for a non-magnetized plasma (Appendix).

A covariant tensor notation is used to provide a unified treatment for electric and magnetic interactions and space-time variations. A further advantage is the natural derivation of interrelationships between different transfer processes, such as pitch-angle scattering and Fermi acceleration (Section 8).

The Fokker-Planck coefficients are evaluated explicitly for two axisymmetric magnetic field models and one isotropic model, all three of which include arbitrary circular polarisation factors. The theory is then applied to pitch-angle scattering, spatial diffusion parallel to the mean field, gradient-induced anisotropies and Fermi acceleration processes. Comparison with observations will be given in later papers.

The random electromagnetic field is regarded throughout as given. Normally, the electromagnetic fluctuations will be associated with plasma waves, whose dispersion
and stability characteristics depend on the particle distributions. To predict both the particle distributions and the electromagnetic spectrum, the Fokker-Planck equation describing the effect of the field on the particle distributions must be considered together with the complementary transport equation describing the back-interaction of the particle distributions on the plasma-wave spectra. However, we shall not treat the full cross-coupled problem here. The question has been discussed for the case of the radiation belts by Kennel and Petschek (1966) and for conditions in interplanetary space by Scarf (1966).

2. Representation of the field and equations of motion

Consider the relativistic motion of a particle of charge $e$ and rest mass $m$ in an electromagnetic field which consists of a uniform magnetic component $B$ in the $x^3$-direction and superimposed random fluctuations.

We assume that the random field has zero mean and is statistically stationary and homogeneous. Its four-potential $\varphi_i$ ($i = 0, 1, 2, 3$) may then be represented by a Fourier-Stieltjes integral

$$\varphi_i(x) = \int d\varphi_i(k) e^{ikx},$$

(2.1)

$$d\varphi_i(-k) = d\varphi_i(k)*,$$

$$i = \sqrt{-1},$$

with statistically orthogonal Fourier increments,

$$\langle d\varphi_i^*(k') d\varphi_j(k) \rangle = H_{ij}(k) \delta(k' - k) dk dk',$$

(2.2)

where $H_{ij}(k)$ is the (four-dimensional) electromagnetic spectral density matrix. Cornered parentheses denote ensemble expectation values.

The Lorentz condition $\partial\varphi_i / \partial x_i = 0$ yields

$$k^i H_{ij} = k^j H_{ij} = 0$$

(2.3)

Covariant and contravariant components are related through

$$x_i = g_{ij} x^j$$

where

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -c^{-2}, \quad g_{ij} = 0 \text{ for } i \neq j.$$

The equations of motion of the particle may then be written

$$\frac{du^i}{d\tau} - \Omega M^i_j u^j = \frac{e}{mc} F^i_j u^j$$

(2.4)
where \( u^i = \frac{d r^i}{d \tau} \), \( r^i \) is the particle position, \( \tau \) is the eigen-time, \( dr^2 = dx^i dx^i \), \( \Omega = eB/mc \) is the cyclotron frequency\(^1\),

\[
M_j^i = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]  

(2.5)

and \( F_{ij} \) is the random electromagnetic field tensor,

\[
F_{ij} = \frac{\partial \varphi_j}{\partial x^i} - \frac{\partial \varphi_i}{\partial x^j} = \int dF_{ij}(k) e^{ik \cdot x} \]  

with

\[
dF_{ij}(k) = t(k_i d \varphi_j(k) - k_j d \varphi_i(k)) \]  

(2.6)

We shall assume that the random field can be regarded as a small perturbation. The zero'th order particle motion \( \overset{\circ}{u}^i, \overset{\circ}{u}^j \) is then given by the equations

\[
\frac{d\overset{\circ}{u}^i}{d\tau} - \Omega M_j^i \overset{\circ}{u}^j = 0 \]  

(2.7)

The general solution is

\[
\overset{\circ}{u}^i = a_j^i \overset{\circ}{u}^j \]  

(2.8)

where

\[
\overset{\circ}{u}^j = U_j^e e^{i \omega_j \tau}, \quad U_j^e = \text{const} \]  

(2.9)

and

\[
a_j^i = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & \frac{\sqrt{2}}{t} & -\frac{\sqrt{2}}{t} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]  

(2.10)

is the eigen-vector matrix of equation (2.7); the eigen-frequencies are

\[
\omega_{(0)} = \omega_{(3)} = 0 \\
\omega_{(1)} = -\omega_{(2)} = \Omega \]  

(2.11)

\(^1\) \( \Omega \) can be positive or negative, according to the sign of the charge.
We note that according to equation (2.9) the amplitudes $U_i^j$ and eigen-frequencies $\omega_i$ are not vector variables. Parentheses are used here to denote indices which are excluded from the sum convention.

The amplitudes satisfy the reality conditions

$$U^1 = (U^2)^*$$

and the relation

$$(U^0)^3 - \frac{1}{c^2}(2|U^1|^2 + |U^3|^2) = 1,$$  \hspace{1cm} (2.12)

which follows from the normalisation $u^t u_t = 1$.

The usual parallel and perpendicular velocities are given by

$$u_\parallel = U^3, \quad u_\perp = \sqrt{2}|U^1|.$$  \hspace{1cm} (2.13)

Besides velocities $u^t, u_\parallel, u_\perp$ with respect to eigen-time we shall use velocities $v^t = dx^i/dt$, $v_\parallel, v_\perp$ with respect to real time, where

$$v^t = u^t / \gamma, \quad \gamma = u_0 = \left(1 - \frac{v_\parallel^2 + v^2_\perp}{c^2}\right)^{-\frac{1}{2}} = \left(1 + \frac{u_\perp^2 + u^2_\parallel}{c^2}\right)^{\frac{1}{2}},$$

represents the ratio of the total energy of a particle to its rest energy.

For future reference, we write down the zero'th order particle position

$$x^0 = (a^0_0 U^0 + a^0_3 U^3) \tau + \frac{a^1_0 U^1 e^{i\Omega \tau}}{i\Omega} - \frac{a^1_2 U^2 e^{-i\Omega \tau}}{i\Omega}.$$  \hspace{1cm} (2.14)

In the following, we shall be concerned primarily with the non-stationary response of the linear system (2.7) to various forms of resonant excitation. For this purpose, we need to resolve the excitation into its normal-mode constituents, i.e. to transform from variables $u^t$ in a cartesian basis to helical variables $u_i$ (non-cursive symbols) in the eigen-vector basis. The transformation is given by equations (2.8), (2.10); it affects only the perpendicular components,

$$u^1 = \frac{1}{\sqrt{2}}(u^1 + u^2), \quad u^1 = \frac{1}{\sqrt{2}}(u^1 - u^2)$$

$$u^2 = \frac{i}{\sqrt{2}}(u^1 - u^2), \quad u^2 = \frac{1}{\sqrt{2}}(u^1 + u^2).$$  \hspace{1cm} (2.15)