Abstract

Reinforced concrete frames under spatial action exhibit distinct nonlinear behaviour. The material nonlinearity can be captured at cross-sectional level. Existing cross-sectional models can be classified into resultant models, truss models, uniaxial models, wall models, as well as models based on finite element analyses. Here, the authors propose a new hybrid approach. The approach combines a uniaxial model with a wall model. The cross section is divided into two components. The stirrups, the adjacent concrete and longitudinal rebars form a notional, thin-walled cross section assembled from membrane elements. The second, uniaxially stressed component comprises the remaining concrete and longitudinal reinforcement. The strain state of the cross section follows from the equations of equilibrium, continuity, and the constitutive laws.

Keywords: reinforced concrete, frame, nonlinear analysis, cross-sectional analysis, hybrid approach.

1 Introduction

Many of the common structures consist of frame elements. New construction methods and progress in concrete technology render possible bolder and more slender concrete structures. Their safe and economical realisation calls for nonlinear and powerful computational methods. Currently used structural analysis software is suitable only to a limited extent for these highly stressed structures. Particularly with regard to spatial frames, the geometrical and material nonlinearities are not accurately considered [1].

Within beam theory the material nonlinearity can be captured at cross-sectional level. A cross-sectional analysis provides the strain state corresponding to a set of section forces. One application of a cross-sectional analysis are interaction diagrams. Interaction diagrams visualise the bearing capacity of a cross section under combined
loading. A cross-sectional analysis can also be applied in the computation of force-strain relationships. A typical example are moment-curvature curves. These curves describe the structural behaviour of a cross section. The main application of a cross-sectional analysis, however, is the nonlinear structural analysis of frames [2].

The authors propose a classification for existing cross-sectional approaches. Resultant models, truss models, uniaxial models, wall models, as well as models based on finite element analysis can be distinguished. Resultant models are not suitable for a general cross-sectional analysis. Truss models apply only to the ultimate limit state. Uniaxial models are generally restricted to biaxial bending and normal force. Wall models assume a locally plane strain and stress state and thus are able to cover transverse forces and Saint-Venant’s torsion. A detailed characterisation and examples for all classes are given in section 2. However, so far there is no generally accepted model for arbitrary cross sections considering all six section forces and the associated interactions. A new approach is presented in section 3. This hybrid approach combines a uniaxial model with a wall model. The equations of equilibrium, continuity and, the constitutive laws are derived. The incorporation of the hybrid cross-sectional approach into a structural analysis program allows for a new, uniform design and checking procedure for concrete structures. For a more detailed discussion of this aspect the reader is referred to [3].

2 Existing cross-sectional approaches

2.1 Resultant models

Resultant models describe the cross-sectional behaviour directly in the form of relationships between internal forces and generalised strain components. These relationships are either completely empirically derived or result from a detailed cross-sectional analysis. In the latter case, the relationships are simply an interpolation between characteristic points obtained from a detailed analysis. A very simple resultant model is applied for the limit analysis using plastic hinges. The static and kinematic approach presume an ideal elastic-plastic moment-curvature relationship for the determination of the upper and lower bounds. Another example for a resultant model is the German concrete code [4] which proposes a trilinear moment-curvature curve for the nonlinear analysis. A unidirectional influence of one internal force on another can be captured by a set of force-strain curves and an associated interpolation method. For a complete interaction of the internal forces the procedure is inappropriate. An alternative approach is based on existing plasticity models [5]. Instead of the principle stresses of a three-dimensional material model, the model is formulated in terms of bending moments and axial forces. However, for each cross section the model parameters have to be calibrated anew. Resultant models are suitable in the scope of a nonlinear structural analysis. However, they are inappropriate for a general and detailed cross-sectional analysis.
2.2 Truss models

The first truss models for the analysis of the structural behaviour of reinforced concrete under bending moment, normal and transverse forces (Figure 1) were developed by Ritter [6] and Mörsch [7, 8]. Rausch [9] extends these models for spatial trusses under torsion (Figure 2). The models are continuously improved and added to. Until now, most concrete codes apply truss models in the design for transverse forces and torque. Truss models satisfy equilibrium but not compatibility. Despite many modifications the use of truss models is therefore restricted to the ultimate limit state.

2.3 Uniaxial models

Uniaxial models assume a simple uniaxial stress and strain state in axial direction of the frame element. Regarding biaxial bending and axial force the cross section is presumed to remain plane (Bernoulli-hypothesis). The determination of the strain state corresponding to a given set of three internal forces is carried out by solving the inverse problem. First the the resultants of a given strain state are computed by cross-sectional integration. Starting with applicable initial values the strain state is improved iteratively. The superordinate iteration is performed using Newton-Raphson method, optimisation, or minimisation algorithm. This type of nonlinear cross-sectional analysis is commonly used in the design of frames and uniaxial slabs in the form of tables, diagrams, and programs.

A wide range of articles on cross-sectional integration is published. An overview is given by [10] and [11]. An analytical integration is only suitable for basic geometry and simplified stress-strain curves. For complex cross-sectional geometries a discretisation and subsequent summation is appropriate. This approach is called fiber model. For instance, trapezium, stripes, or triangulares act as discretisation elements. The stresses of the elements are integrated analytically, numerically, or combined analytically-numerically. Alternatively, the double integral over the cross
Table 1: Membrane models.

<table>
<thead>
<tr>
<th>Name</th>
<th>Crack</th>
<th>Angle</th>
<th>Feature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compression Field Theory [15]</td>
<td>smeared</td>
<td>strain</td>
<td>considering compatibility</td>
</tr>
<tr>
<td>Modified Compression Field Theory [16]</td>
<td>mixed</td>
<td>strain</td>
<td>modif. concrete tension curve</td>
</tr>
<tr>
<td>Cracked Membrane Model [17]</td>
<td>local</td>
<td>strain</td>
<td>local equilibrium condition</td>
</tr>
<tr>
<td>Disturbed Stress Field Model [18]</td>
<td>mixed</td>
<td>both</td>
<td>diff. stress and strain angle</td>
</tr>
<tr>
<td>Fixed Angel Softened Truss Model [19]</td>
<td>smeared</td>
<td>stress</td>
<td>consis. shear transfer model</td>
</tr>
<tr>
<td>Softened Membrane Model [20]</td>
<td>smeared</td>
<td>stress</td>
<td>considering Poisson’s ratio</td>
</tr>
</tbody>
</table>

The section is transformed by Green’s theorem into a line integral around the border of the cross-section. Thus only one-dimensional functions have to be integrated on the edges. In this context sometimes also Gauss’s theorem or Cauchy’s theorem are referred to. However, all three theorems are based on Stokes’ theorem. In [12] the transformation is presented for polygonal cross-sections and analytical line integration, in [13] the remaining integration is performed numerically. The approach of [14] differs from the other uniaxial models. In addition to biaxial bending and normal force, warping torsion is considered. This approach is restricted to thin-walled sections and neglects Saint-Venant’s torsion, as well as transverse forces. Regarding the common thick-walled, closed thin-walled, and solid cross sections in concrete structures, the warping torsion is circumstantial. Therefore uniaxial models are primarily appropriate for biaxial bending with normal force.

2.4 Wall models

Wall models discretise the cross section into coupled wall elements. In each wall element a membrane stress state is assumed. Compared to the uniaxial models, transverse forces and Saint-Venant torsion can additionally be captured. Beside the generalised strain values of the frame element, the local stress and strain state of the respective membrane element has to be determined.

For reinforces concrete membrane elements a wide range of various models exist. Membrane elements are only subjected to in-plane forces. Thus, two normal stresses and one shear stress and the three corresponding strain values result. Compared to truss models, membrane models additionally consider compatibility. The uniaxial constitutive laws for concrete are specified in a rotated reference system. The angle is either defined by the principle strain or by the principle stress direction. The models mostly apply smeared crack, but also local approaches. An overview of important membrane models is given in Table 1.

The model proposed by Vecchio & Collins [21] uses the MCFT to analyse plane reinforced concrete frame elements subjected to bending moment, normal and transverse force. The cross section is modeled by membrane elements arranged one upon the other (Figure 3). In [22] a model for spatial frame elements with rectangular and T-shaped cross-sections is presented. The outer area is modeled by membrane elements
following the CFT, the inner concrete core is neglected (Figure 4). For the distribution of the shear stresses produced by transverse forces, a simplified approach is used to determine an average shear strain of the frame. Cocchi & Volpi [23] propose a model for arbitrary polygonal cross sections under torque, biaxial bending and normal force. Transverse forces are not considered. Only the border of the cross section is modeled applying the MCFT, the inner core is again neglected (Figure 5).

An approach differing from the ones mentioned above is presented by Rahal & Collins [24] for rectangular cross sections. The cross section is notionally divided into two interacting systems. In the first system, consisting of the concrete section and the longitudinal reinforcement, a uniaxial stress state is assumed. The second system is assembled from membrane elements following the MCFT. The elements form a thin-walled hollow section with the outside dimension identical to the original section (Figure 6). The transverse forces are uniformly distributed to the walls, and an average shear strain of the frame is determined.

The model proposed by Hartung & Krebs [25] is suitable for the analysis of polygo-
nal cross sections. The whole cross section is discretised into wall elements (Figure 7). In contrast to the models mentioned before, the walls can include more than one material point. The model is the only one computing the shear distribution produced by the transverse forces under consideration of both equilibrium and compatibility. It captures the local shear strains, but presumes the shear stiffness of the frame element as a whole to be infinite. The above-mentioned models neglect the transverse forces or oversimplify the resulting shear distribution. A disregard of the transverse forces is in particular problematical when torque acts concomitantly, and the shear stresses superpose each other. Furthermore the bending and axial strength, and the stiffness reduction caused by shear forces are ignored. However, the model of [25] is based on simplified constitutive laws, because it neglects the shear transfer across cracks. Furthermore, the softening influence of the perpendicular tension to the compression strut has to be considered in advance by scaling of the constitutive compression law. The model also disregards the influence of warping in the membrane elements due to bending and torsion. Warping of the membrane elements leads to a curvature in the assumed concrete strut. All above-mentioned models for spatial frame elements consider the warping by dint of an equivalent thickness of the membrane element.

2.5 Finite element analysis

The finite element method (FEM) has become a universal analysis tool in structural mechanics. In the analysis of reinforced concrete frames also two- and three-dimensional elements are applied. For instance, Karayannis [26] uses two-dimensional elements for the solution of a modified Poisson’s equation within the Saint-Venant torsion theory. The solution is restricted to plain concrete sections without reinforcement. For concrete a simplified bilinear stress-strain curve is assumed. The material failure criteria considers the influence of section forces in addition to torque. In [27] the cross-section is discretised into three different types of elements. Depending on the position of the element in the cross section, uniaxially stressed elements, elements considering one normal and one shear stress, and elements considering one normal and two shear stresses are used. In a comparative study Ghavamian & Carol [28] analyse three-point
Figure 8: Division of the cross section in the scope of the hybrid approach.

bending beams applying various types of three-dimensional finite elements. While a failure due to bending can be acceptably well captured by most of the tested models, the numerical results of shear failure show clear discrepancies.

On the one hand the finite element analysis needs no assumptions concerning kinematic and stress distribution, on the other hand the hypotheses of beam theory are highly appropriate. The use of frame elements reduces the number of degrees of freedom, and hence the computational cost. In addition, the results are directly available in the form of generalised strains and internal forces, which are more convenient and better understood by practicing engineers than stress or strain values. Modeling a whole frame structure with three-dimensional elements raises difficulties of defining boundary conditions and reinforcement, and requires an adequate discretisation.

3 Hybrid approach

3.1 Basic concept

The new hybrid approach combines the extended opportunities of a wall model with the advantages of a simple uniaxial model. The model considers all six internal forces of a spatial frame element with an arbitrary cross section. The basic concept is a notional division of the cross section into two types of areas. It is presumed that a biaxial stress state only arises where stirrups and longitudinal rebars form a reinforcement net. These areas are modeled by membrane elements with a uniform stress distribution. In Figure 8 they are depicted in red colour. The membrane elements are assumed to be thin-walled. The warping of the biaxially stressed area is not restricted. In the remaining, blue coloured areas a uniaxial stress state is assumed. While the section assembled from membrane elements contributes to all six internal forces, the uniaxial areas are confined to the two bending moments and the normal force.

The strain state is specified by generalised strain values and the local strains of the membrane elements $\varepsilon_l$, $\varepsilon_t$ and $\gamma_{ll}$. The generalised strain values consist of the elongation $\varepsilon_0$, the curvatures $\kappa_y$ and $\kappa_z$, the twist $\kappa_x$ and the axial derivatives of the elongation $\varepsilon'_0$, and the curvatures $\kappa'_y$ and $\kappa'_z$. A shear deformation of the frame element is neglected. Additionally, the redundant shear force of the closed biaxial section has
to be determined. The required equations are ordered according to Navier’s three principles, which are compatibility (Section 3.2), equilibrium (Section 3.3) and the constitutive law (Section 3.4).

### 3.2 Compatibility equations

#### 3.2.1 Compatibility of membrane strains

The constitutive laws for concrete in membrane elements are specified in a \(dr\)-coordinate system rotated by \(\theta\). For this reason a transformation of the three local strain values is required

\[
\begin{bmatrix}
\varepsilon_d \\
\varepsilon_r \\
\gamma_{dr} \\
\end{bmatrix} =
\begin{bmatrix}
\cos^2 \theta & \sin^2 \theta & 2 \sin \theta \cos \theta \\
\sin^2 \theta & \cos^2 \theta & -2 \sin \theta \cos \theta \\
-\sin \theta \cos \theta & \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \\
\end{bmatrix} \cdot
\begin{bmatrix}
\varepsilon_l \\
\varepsilon_t \\
\gamma_{lt} \\
\end{bmatrix}.
\]

(1)

The \(l\)-axis of the original \(lt\)-coordinate system is parallel to the \(x\)-axis of the cross-section. The inverse transformation can be performed with the transposed rotation matrix.

#### 3.2.2 Crack angle in membrane element

The assumed crack angle \(\theta\) is identical to the assumed direction of the concrete strut. A common definition of the angle is the principal strain direction

\[
\tan 2\theta = \frac{\gamma_{lt}}{\varepsilon_l - \varepsilon_t}.
\]

(2)

Thus, the shear strain \(\gamma_{dr}\) disappears in (1). Another approach is based on the principal stress direction of the three applied boundary stresses of the membrane element

\[
\tan 2\theta = \frac{2\tau_{lt}}{\sigma_l - \sigma_t},
\]

(3)

whereas this relationship strictly speaking belongs to the equilibrium equations (Section 3.3). The inverse tangent is ambiguous. Thus, the two principle directions are ordered taking into account

\[
\varepsilon_d < \varepsilon_r.
\]

(4)

#### 3.2.3 Compatibility of axial strain

The compatibility of axial strain for both the uniaxial and the biaxial area is ensured by Bernoulli’s hypothesis. The axial strains form a plane and are determined for each point by

\[
\varepsilon_x = \varepsilon_0 + \kappa_y z - \kappa_z y
\]

(5)

For membrane elements the axial strain \(\varepsilon_l\) follows from the strain at its centroid. Due to the additional shear the biaxial cross section warps.
For torque the membrane elements form a closed hollow section, in which the continuity has to be fulfilled. Hence, a notional cut in the cross section should not gape. Similar to Bredt’s torsional formula the relative displacement at the section ends follows from a closed loop integral
\[
\Delta u = \oint \gamma_t\, dq - \kappa_x\, A_m = 0.
\] (6)
where \(A_m\) is the area enclosed by the centerline of the section (Figure 9).

### 3.2.5 Warping of the membrane elements

The assumed uniform and plane stress distribution of a membrane element imply an approximation. Due to torque and bending moments the membrane elements curve. The curvature of the assumed concrete strut \(\kappa_r\) can be derived geometrically \[3\] and results in
\[
\kappa_r = -\kappa_x \sin 2\theta - (\kappa_y \cos \omega + \kappa_z \sin \omega) \cos^2 \theta,
\] (7)
where \(\omega\) defines the orientation of the membrane element (Figure 9). Most of the existing wall models consider the curvature of the concrete strut by an equivalent thickness of the membrane element. Thereby, an inconsistence arises for cross sections without curvature and twist. In this case the membrane thickness reduces to zero. On account of this, the authors propose as an alternative approach a modification of the constitutive law of the concrete strut. An adequate relationship, comparable with the influence of the perpendicular tension, is currently being derived.
3.3 Equilibrium equations

3.3.1 Local equilibrium in membrane element

For the membrane elements a plane, uniform stress state is assumed. The reinforcement is arranged parallel to the local coordinate axis. The stresses at the edges of the membrane elements result from the reinforcement stresses $\sigma_{sl}$ and $\sigma_{st}$, as well as from the concrete. The contribution of concrete consists of the stress of the concrete strut $\sigma_d$, the perpendicular tension stress $\sigma_r$, and in general of the shear stress $\tau_{dr}$. The transformation is similar to (1) and results in

$$
\begin{bmatrix}
\sigma_l \\
\sigma_t \\
\tau_{lt}
\end{bmatrix}
= 
\begin{bmatrix}
\cos^2 \theta & \sin^2 \theta & -2 \sin \theta \cos \theta \\
\sin^2 \theta & \cos^2 \theta & 2 \sin \theta \cos \theta \\
\sin \theta \cos \theta & -\sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta
\end{bmatrix}
\cdot
\begin{bmatrix}
\sigma_d \\
\sigma_r \\
\tau_{dr}
\end{bmatrix}
+ 
\begin{bmatrix}
\rho_{sl}\sigma_{sl} \\
\rho_{st}\sigma_{st} \\
0
\end{bmatrix}, \quad (8)
$$

where $\rho_{sl}$ and $\rho_{st}$ represent the percentage of reinforcement in the corresponding direction.

3.3.2 Equilibrium for biaxial bending and normal force

The stress resultants of the cross section have to comply with the given bending moments and normal force. The integration of the normal stresses is carried out for the uniaxial area, for the membrane elements and the remaining longitudinal reinforcement

$$
\begin{align*}
N &= \sum N_{\text{uniax}} + \sum N_{\text{mem}} + \sum N_{\text{rein}}, \\
M_y &= \sum M_{y,\text{uniax}} + \sum M_{y,\text{mem}} + \sum M_{y,\text{rein}}, \\
M_z &= \sum M_{z,\text{uniax}} + \sum M_{z,\text{mem}} + \sum M_{z,\text{rein}}.
\end{align*} \quad (9, 10, 11)
$$

For the integration of the uniaxial area the approach of [13] is applied. Since a uniform stress state is assumed for the respective membrane elements, the resultant normal force is the product of the normal stress $\sigma_l$ and the membrane area $hb$. The contribution to the bending moments arises according to the corresponding lever arm. The same applies to the remaining reinforcement.

3.3.3 Equilibrium for transverse forces

Also the transverse forces have to correspond to the stress resultant of the cross section. Within beam theory transverse forces comply with the longitudinal derivatives of the bending moments. The derivatives follow from the stress resultant of the previous subsection. The derivatives require the gradient curves of the applied constitutive laws. For a complete derivation including the trigonometric functions and constitutive laws the reader is referred to [30]. In the hybrid model, the transverse forces only
result from the stress of the biaxial section. In the uniaxial section the shear forces are neglected for the sake of simplicity

\[ M'_y = V_z = \sum M'_{y,\text{mem}} = \sum z_{\text{mem}} N'_{\text{mem}}, \quad (12) \]
\[ M'_z = V_y = \sum M'_{z,\text{mem}} = \sum -y_{\text{mem}} N'_{\text{mem}}, \quad (13) \]

### 3.3.4 Equilibrium for torsion

The torque resulting from the shear stresses according to Saint-Venant torsion have to be concordant with the given torque. The contribution of a membrane element consists of the product of area \( hb \), shear force \( \tau_{lt} \), and lever arm \( r_\perp \)

\[ M_x = \sum r_\perp h b \tau_{lt}. \quad (14) \]

The shear centre, the centre of torsional rotation, and the axis of the frame element are presumed as congruent (Figure 9).

### 3.3.5 Equilibrium in biaxial area

The forces acting on the membrane elements are determined by successive intersecting of the biaxial area (Figure 10). For each membrane element, the shear force \( T \) and the transversal normal force \( n_t \) follow from the equilibrium condition in axial direction and in local transversal direction, respectively

\[ T_m = -T_0 - \sum N'_x \quad \text{and} \quad n_t = 0. \quad (15) \]

The determined forces have to correspond to the resultant stresses \( \sigma_t \) and \( \tau_{lt} \) of the membrane elements. Since only the boundary forces are known an average value is calculated for the membrane stresses. Within the hybrid approach the shear distribution produced by transverse forces thereby conforms with equilibrium, compatibility and the constitutive laws.

### 3.4 Constitutive laws

A fully smeared crack approach is applied, i.e. average stresses and strains are regarded. The constitutive laws specify the relationship between stress and strain and thus form the link between the compatibility and equilibrium equations. All material formulations are based on the FASTM [19]. The stress-strain relationship for the concrete compression strut is given by

\[
\sigma_d = \begin{cases} 
\zeta f_c \left[ 2 \left( \frac{\varepsilon_d}{\zeta \varepsilon_{c1}} \right) - \left( \frac{\varepsilon_d}{\zeta \varepsilon_{c1}} \right)^2 \right] & \text{for } \left( \frac{\varepsilon_d}{\zeta \varepsilon_{c1}} \right) \leq 1 \\
\zeta f_c \left[ 1 - \left( \frac{\varepsilon_d/(\zeta \varepsilon_{c1}) - 1}{4/\zeta - 1} \right)^2 \right] & \text{for } \left( \frac{\varepsilon_d}{\zeta \varepsilon_{c1}} \right) > 1 
\end{cases} \quad (16)
\]
where $\varepsilon_{c1}$ represents the strain at stress peak $f_c$ (Figure 11). The softening influence of the perpendicular tension is characterised by $\zeta$ according to

$$\zeta = \frac{0.9}{\sqrt{1 + 400\varepsilon_r}}.$$  

(17)

For the behaviour under tension (Figure 12) a linear function with subsequence softening is defined

$$\sigma_r = \begin{cases} 
E_c\varepsilon_r & \text{for } \varepsilon_r \leq \frac{f_{ctm}}{E_{cm}} \\
 f_{ctm} \left( \frac{f_{ctm}}{E_{cm}\varepsilon_r} \right)^{0.4} & \text{for } \varepsilon_r > \frac{f_{ctm}}{E_{cm}}.
\end{cases}$$

(18)

Both compression and tension curves are also applied for the uniaxial area. A simplified shear modulus is derived in [19] based on the existing stress-strain curves

$$\tau_{dr} = \frac{\sigma_d - \sigma_r}{2(\varepsilon_d - \varepsilon_r)} \gamma_{dr}.$$  

(19)

For the reinforcement, a typical bilinear stress-strain relationship is assumed, whereat the influence of local yielding at a crack is taken into account.

3.5 Solution strategy

Due to the nonlinear material behaviour a direct determination of the strain state is not possible, so that an iterative solution becomes necessary. The number of unknowns,
and hence the computational cost, is reduced by converting and inserting the equations derived above. The remaining equation system is partitioned depending on the interactions of the equations. The stress state is determined in four levels. At the first level the curvature and elongation are computed. The second level provides the derivatives of these three values. The torsional unknowns — the twist and the initial shear force in the closed hollow section — are determined at the third level. At the last level the local membrane strains are computed. At each level a damped Newton-Raphson iteration is applied. Most of the required Jacobian matrices are derived analytically. The remaining matrices are computed using difference quotients.

4 Conclusions

The material nonlinearity of reinforced concrete frames can be captured on cross-sectional level. A cross-sectional analysis determines the stress and strain distribution corresponding to the internal forces. A classification of existing approaches for the cross-sectional analysis is proposed. The proposed classes are composed by resultant models, truss models, uniaxial models, wall models as well as models based on finite element analyses. Beside the classification, a new hybrid approach is developed. The new approach combines the extended opportunities of a wall model with the advantages of a simple uniaxial model. The equations of equilibrium and continuity, and the constitutive laws are presented.

References