The author compared real-number and complex-number expressions of aerodynamic forces for flutter analysis and concluded that a complex expression is more compact in format, more efficient in computation, and easier to understand from an educational viewpoint than a real-number expression.

In flutter analysis, to solve an equation of motion derived from a real-number expression of aerodynamic forces, the solution of motion is typically assumed to have the form

$$u = Ae^t = Ae^{ikt}$$

where $u =$ motion; $A =$ constant; $\delta =$ logarithmic decrement; $k =$ reduced frequency $= \omega BU; \omega =$ circular frequency of motion; $B =$ reference length; and $U =$ wind speed.

The nature of the motion depends on the value of the logarithmic decrement, $\delta$. More specifically, $\delta < 0, \delta = 0,$ and $\delta > 0$ correspond to decreasing-amplitude (converging), constant-amplitude (onset of instability), and increasing-amplitude (diverging) sinusoidal motions, respectively. By keeping $\delta$ as a variable in the solution process, the behavior of a bridge can be investigated in its preflutter ($\delta < 0$), onset of flutter ($\delta = 0$), and postflutter ($\delta > 0$) stages.

While it is beneficial to keep $\delta$ variable to better understand bridge response in the different stages, $\delta$ need not be part of the solution. If one is only interested in the onset of flutter, $\delta$ can be set equal to zero. As noted by the author, the equation of motion derived from a real-number expression of aerodynamic forces then reduces to (11), which is as compact as (10), derived from a complex-number expression. Furthermore, by setting $\delta = 0$, the independent number of variables in the equation of solution written in terms of a real-number expression reduces to two ($\omega$ and $U$ or $\omega$ and $k$)—the same number of unknowns as in a complex expression. Therefore, there are really no significant differences between real- and complex-number expressions in terms of format compactness and computation efficiency.

In the discussers' opinion, the choice of using a real or complex number format should depend on whether the flutter derivatives or complex coefficients are more conveniently measured or predicted. This depends on the available testing facilities and the expertise of the research team. From an educational viewpoint, civil engineers in the United States are indeed more familiar with real than complex numbers.

**Closure by Uwe Starossek,**

Member, ASCE

The writer appreciates the discussers' interest in his paper and welcomes the opportunity to provide further clarification and conclusions.

1. The discussers state that the logarithmic decrement $\delta$, in the real-number flutter algorithm, can be set equal to zero if one is only interested in the critical stage of flutter onset. Based on this fact, they argue that "there are really no significant differences between real- and complex-number expressions in terms of format compactness and computation efficiency.'

The scope of the paper has indeed been limited to the special but practically important problem of flutter onset and the prediction of the critical wind speed. The validity of the references used for comparing real and complex notation, however, is also limited to this special problem (as shown below). The paper's findings are therefore believed to be conclusive and valid. Still, by discussing noncritical states of vibration, an interesting topic has been raised which, when discussed carefully, reveals further strong substantiation of the conclusions reached in the paper.

Eigenvalue problem (8), derived from real-number aerodynamic forces, can be reduced to (11), which is as compact as (10), derived from complex-number expressions. The discussers point out that this simplification is possible only for the critical stage ($\delta = 0$). For a study of undercritical (converging) or overcritical (diverging) states of vibration, one must revert to a more complicated eigenvalue problem such as (8). Such a reversion to a more general and more complicated equation is unnecessary, however, when using complex-number expressions. In that case, the eigenvalue problem as stated in the simple form (10)—and, in fact, almost the entire analysis algorithm—already cover the more general case of noncritical vibrations. The only difference to the critical stage is that now the reduced frequency $k$ is understood to be a complex quantity. In other words, the real-number equivalent of (10) would be (8) and not (11). Thus, the advantages of complex notation in terms of clarity and format compactness become particularly apparent when extending the study to noncritical stages.

Some additional remarks concerning noncritical states of vibration are in order. When introducing $k$ as a complex quantity, it follows immediately that the unsteady aerodynamic coefficients, or flutter derivatives, are also to be determined as functions of this complex parameter. While the required effort might appear prohibitive when using experimental techniques, an analytical or numerical determination of aerodynamic coefficients for complex, instead of real, $k$ would not pose a significant additional problem. Evaluation of Theodorsen's function $C(x)$ for complex $k$, for instance, has been reported by Edwards et al. (1977).

The real-number equivalents of complex $k$ are nonzero $\delta$. Hence, for a real-number description of noncritical stages, the flutter derivatives should be determined as functions of the two real parameters $k$ and $\delta$. Such an approach has not been pursued by Namini et al. (1992)—one of the references used in the paper—when attempting to analyze noncritical states of vibration. Thus, their analysis is valid only for the critical state of constant-amplitude vibration, and keeping $\delta$ variable is of little use. An analysis of noncritical stages based on flutter derivatives that are functions of only one real parameter $k$ can at best be of indicative value and only for stages that are not too far from the critical state of vibration. If such a pseudo-noncritical stage analysis is desired, it can also, and more easily, be performed by exploring the complex-number expression (10), as shown in Starossek (1991).

The paper's comparison of real and complex notation
is limited to the expressions for aerodynamic forces and the equations stating the eigenvalue problem. When considering other details of practical flutter analysis, further substantiation of the paper’s findings is found. A review of finite-element aeroelastic matrices presented in real notation (Namini 1991) and in complex notation (Sta-rossek 1991, 1993), for instance, reveals the advantages of the complex-number approach, particularly in terms of format compactness and simplicity. In fact, finite-element matrices in real notation grow so voluminous and irregular that even routine tasks such as typesetting and programming become a challenge.

2. The discussers state that “the choice of using a real or complex number format should depend on whether the flutter derivatives or complex coefficients are more conveniently measured or predicted” and that this “depends on the available testing facilities and the expertise of the research team.”

The relationship between real flutter derivatives and complex aerodynamic coefficients is given by (7). Even if the flutter derivatives have been measured in real-number format, the corresponding complex coefficients are readily obtained from these simple equations. Thus, different testing facilities and special expertise are not required for determining complex coefficients.

In case “available testing facilities” and “expertise of the research team” refers to the difference between the free-oscillation method and the driven-oscillation method for measuring unsteady aerodynamic forces, it should be noted that both the real-number derivatives and the complex-number aerodynamic coefficients are usually understood to refer to the critical state of constant-amplitude vibration. In both cases, therefore, only a constant-amplitude procedure such as the driven-oscillation method can, in principle, give accurate results, and both sets of coefficients are related by (7).

3. Finally, the discussers state that “from an educational viewpoint, civil engineers in the United States are indeed more familiar with real than complex numbers.”

Real notation in bridge flutter analysis has become customary in the United States. However, the term “educational standpoint” is used in the paper in quite a different sense: not to justify what has become customary but to illustrate the usefulness of complex flutter analysis in terms of simplicity and ease of understanding.

APPENDIX. REFERENCES


Simplified Model of Low Cycle Fatigue for RC Frames

Discussion by Alberto Carnicero, Ricardo Perera, and Enrique Alarcón

To model strength degradation due to low cycle fatigue, at least three different approaches can be considered. One possibility is based on the formulation of a new free energy function and damage energy release rate, as was proposed by Ju (1989).

The second approach uses the notion of bounding surface introduced in cyclic plasticity by Dafalias and Popov (1975). From this concept, some models have been proposed to quantify damage in concrete or RC (Suaris et al. 1990). The model proposed by the author to include fatigue effects is based essentially in Marigo (1985) and can be included in this approach. In the formulation of the fatigue law, the loading-unloading irreversibility concept is employed. However, in this model, an additional constant α is introduced. The physical meaning of this constant is not clear, and neither is its evaluation.

In the approach developed by the discussers, the classical concepts of yield and damage domains are used and the traditional notion of damage energy release rate as a variable related to the elastic strain energy is kept.

For it, the dissipative potentials proposed by the author (1995) are slightly modified to introduce cumulative effects. The discussers propose the following function:

\[ g = Y - [Y_0 + Z(D)\xi(\omega)] \leq 0 \]  \hspace{1cm} (5)

for the damage dissipative potential. In order to keep the coupling between the damage and the plastic domain, a modification of the plastic dissipative potential is also required:

\[ f = |M - X(D)| - [M_0 - R(D)\sqrt{\xi(\omega)}] \leq 0 \]  \hspace{1cm} (6)

where \( \omega \) is a cumulative parameter.

The fatigue function, \( \xi(\omega) \), must satisfy two restrictions:

\[ \xi(\omega) = 1 \iff \omega \leq \omega_{\text{max}} \]
\[ \xi(\omega) = 0 \iff \omega > \omega_{\text{max}} \]  \hspace{1cm} (7)

This new term introduces a softening effect in the two functions in order to include the fatigue effects.

The discussers propose as fatigue function

\[ \xi(\hat{\theta}, \theta) = 1 - \left[ \frac{\hat{\theta}}{N_r(\hat{\theta}) \cdot \theta} \right]^{10^n} = 1 - \left[ \frac{n}{N_r(\hat{\theta})} \right]^{10^n} \]  \hspace{1cm} (8)

where \( \hat{\theta} \) and \( \theta \) represent the total cumulative rotation and the total rotation (semiamplitude loop), respectively; and \( N_r(\hat{\theta}) \) is number of cycles to failure at \( \hat{\theta} \), semiamplitude. Therefore, the relation between \( \theta \) and \( \hat{\theta} \) is a measure of the number of cycles, \( n \). The most important influence of the higher cycles over the lower ones is considered across the ductility \( \mu \).


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