Source-Seeking with Flocking Glowworm Optimization

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Abstract—In this paper we present a Flocking Glowworm Optimization (FGO) technique - a solution for the problem of unknown, time-varying source seeking scenario in complex environment. Motivated from glowworm swarm optimization and the flocking framework, we present a unified protocol navigating a group of dynamic agents in search for the source. The research introduced a broad theorem using a smooth potential function with finite cut-off and virtual agents. Our work, as well as many others (e.g. [8]–[11]), is based on these results. A straight-forward solution one can propose is to add the flocking element with GSO. Although this solution might be suitable, a composition of two independent forcing elements causes rapid perturbation, poor performances and tuning difficulties. Thus, in this paper we propose a unified solution namely, the Flocking Glowworm Optimization (FGO). This embed the attractive relations between neighbor agents inside the flocking smooth potential function. We receive a single forcing element responsible for both cohesion/separation (flocking) and objective (GSO). The modification is set in adjusting the required relative distance between two neighbor agents. Our proposed algorithm is designed to solve the source-seeking problem, where in a presence of a substance (source) with different concentration levels, the swarm is required to locate and monitor the highest concentration level (i.e field’s maximum). The field is often described as a potential function and considered as an oil spill or toxic cloud which makes this problem relevant and essential to solve. Solutions for the source-seeking problem based on combination of formation control and gradient climbing are varied (e.g. [12]–[16]). In addition to the benefit of avoiding formation control, gradient estimation is a complex process involves errors and enforced restrictions on swarm topology. By setting the objective function, which induces agent’s attractiveness, to the field’s measured concentration level, we receive a cooperative swarm behavior navigating to field’s maxima.

The rest of the paper is organized as follows. In section II and III, the problem statement and preliminaries results are introduced. Section IV describes the Flocking Glowworm Optimization technique. Section V presents simulation of a 3-D problem and an experimental results for a group of 2-D wheeled robots. Finally, in section VI conclusions are drawn.

II. PROBLEM STATEMENT

Consider a multi-agent system of N mobile, autonomous agents and let \( G = (\mathcal{V}, \mathcal{E}) \) be an undirected graph where the communication topology is represented by the adjacency matrix

\[
A = [a_{ij}] \in \mathbb{R}^N \times N
\]

\[
a_{ij} = \begin{cases} 
1, & (i, j) \in \mathcal{E} \\
0, & \text{else} 
\end{cases}
\]

The Laplacian matrix \( L = \Delta - A \), where \( \Delta = \text{diag}(A \cdot 1) \) is the degree matrix of \( G \) with diagonal elements of \( d_i = \sum_{j \in V} a_{ij} \).
\[ \sum_j a_{ij} \] and \(1 = [1, \ldots, 1]^T \in \mathbb{R}^N\). Each agent (node) \(i\) corresponds to the equations of motion,

\begin{align}
\dot{q}_i &= p_i \\
\dot{p}_i &= u_i,
\end{align}

where \(q_i, p_i, u_i \in \mathbb{R}^m\). Let \(\mathcal{N}_i\) be the set of neighbors of agent \(i\)

\[ \mathcal{N}_i = \{ j \in \mathcal{V} : ||q_j - q_i|| < r_s \}, \tag{3} \]

where \(r_s\) is the radial sensor. Let \(d\) be the required (constant) distance between two neighbor agents which define the set of algebraic constraints

\[ ||q_j - q_i|| = d, \forall j \in \mathcal{N}_i. \tag{4} \]

**Definition 1** [7]: An \(\alpha\)-Lattice is a configuration \(q\) satisfying the set of constraints in (4). Let \(\psi(q_i(t)) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^+\) be a spatial constraint level of a bounded substance (e.g. oil spill, toxic cloud).

**Assumption 1**: The field \(\psi\) is proper, i.e. \(\forall \nu \in \mathbb{R}\) the sublevel set \(\{q | \psi(q) \leq \nu\}\) is compact, and has a single maxima at \(q_s\).

Let \(d_s = ||q_s - q_i||\) be the distance between swarm center of mass \((q_s = \frac{1}{N} \sum_{i=1}^{N} q_i)\) and field’s maxima \(q_s\). Then, the swarm objective is to track the highest concentration \(\psi(q_s) = \psi_{\text{max}}\) by performing flocking behavior. In other words, we can use the following definition.

**Definition 2**: An \(\alpha\)-Lattice is said to be located at the source \(q_s\), if

\[ q_s \in C = \left\{ \sum_{i=1}^{N} \beta_i q_i : (\forall i : \beta_i \geq 0) \land \sum_{i=1}^{N} \beta_i = 1 \right\}. \tag{5} \]

### III. PRELIMINARIES

To construct a differentiable (smooth) flocking potential function and weighted adjacency matrix, we use the \(\sigma\)-norm described in [7]. Let \(\sigma\)-norm be a nonnegative map \(\mathbb{R}^m \to \mathbb{R}_{\geq 0}\) defined as

\[ ||z||_\sigma = \frac{1}{\epsilon} \left( \sqrt{1 + \epsilon ||z||^2} - 1 \right), \epsilon > 0 \]

with a corresponding gradient of

\[ \sigma'(z) = \nabla ||z||_\sigma = \frac{z}{\sqrt{1 + \epsilon ||z||^2}} = \frac{z}{1 + \epsilon ||z||_\sigma}. \tag{7} \]

For construction of smooth potential functions with finite cut-offs and smooth adjacency matrices, we use bump function [17] defined by

\[ \rho_h(z) = \begin{cases} 
1 & , z \in [0, h) \\
\frac{1}{2} (1 + \cos(\pi \frac{z - h}{1 - h})) & , z \in [h, 1] \\
0 & , else
\end{cases}, \tag{8} \]

where \(h \in (0, 1)\) is a tuning parameter which determines the slope. We use an algebraic sigmoid based function for describing the repulsive/attractive force behavior

\[ \phi_s(z) = \frac{1}{2} \left[ \frac{(a + b)(z + c)}{1 + (z + c)^2} + (a - b) \right], \tag{9} \]

where \(0 < a \leq b, c = |a - b|/\sqrt{ab}\). From graph theory, the Laplacian matrix \(L\), under undirected graph topology is always positive, semidefinite matrix which satisfies

\[ z^T L z = \frac{1}{2} \sum_{i,j \in E} a_{ij} (z_j - z_i)^2 \tag{10} \]

and its second eigenvalue \(\lambda_2(L)\) determines the speed of convergence (also known as the Fiedler value).

### IV. FLOCKING GLOWWORM OPTIMIZATION

#### A. Algorithm

In this section we present the details of our Flocking Glowworm Optimization (FGO) technique. First we set the luciferin rule to be our sensed concentration level

\[ l_i = \psi(q_i(t)). \tag{11} \]

Here we choose to use a simple version of the luciferin by replacing the cost function in [5] with \(\psi\). Current work involves dynamic luciferin level but the general idea holds.

As described before, the proposed GSO technique is based on applying force in the direction of the most attractive agent \(\forall j \in \mathcal{N}_i\). This causes discontinuities and switching topologies which makes the analysis not differentiable. Our proposition is to integrate this rule inside the flocking potential function. Let \(d_s = ||d||_\sigma\) and let \(q_{ji} = ||q_j - q_i||_\sigma \in \mathbb{R}_{\geq 0}\).

The set of constraints described in (4) can be rewritten as

\[ q_{ji} = d_s, \forall j \in \mathcal{N}_i, \tag{12} \]

then, the unified action function navigating the agents toward the source by enforcing flocking behavior (cohesion, separation) and relative luciferin level, with respect to agent \(j\) on \(i, \phi^\alpha_{ji} \in \mathbb{R}^{2m} \to \mathbb{R}\) is

\[ \phi^\alpha_{ji} = \rho_l(q_{ji}/r_\alpha) \phi_s(q_{ji} - d_l) \]

\[ d_l^\alpha = d_{l_s} - l_{ji} - l_{ji}(q). \tag{13} \]

where \(r_\alpha = ||r||_\sigma\). (for the reset of the paper \(\phi_t = \phi^\alpha_{ji}, d_l = d_l^\alpha\)). For the simple case where \(d_l = d_{l_s}\) (constant) we get the potential function proposed in [7]. The term \(l_{ji}\) express the attractive relations between two neighbor agents,

\[ l_{ji} = (l - l_t) \frac{d_{l_s}}{\gamma l_{\text{max}}}, \tag{14} \]

where \(l_{\text{max}}\) is the maximum luciferin level and \(\gamma > 0\) is a tuning parameter. The element \(d_{l_s}/l_{\text{max}}\) normalized \(l_{ji}\) so that the varying \(d_l\) in (13) keeps the repulsive/attractive structure. In other words if \(|l_{ji}| > d_{l_s}\) then one can get irrational behavior such as exceeding the communication range \(r_s\), collision of agents by cancellation of the repulsive element \((d_{l_s} - l_{ji}) = 0\) or positive feedback. Thus, the constraints on tuning \(\gamma\) are set as

\[ \begin{cases} 
1 < \gamma < d_{\alpha} & , d_{\alpha} > 1 \\
1 < \gamma < \frac{1}{d_{\alpha}} & , d_{\alpha} > 1.
\end{cases} \tag{15} \]

The function \(\phi_t\) in (13) embodies two of Reynolds rules - cohesion and separation when applying attraction/repulsion force with respect to \(q_{ji} - d_l\). This function is depicted
in Figure 1, where if \( l_j > l_i \) then the attraction zone increases and vice versa. For the case when \( l_j = l_i \), luciferin consensus is achieved, and once condition (12) holds then global equilibrium is reached and both source-seeking and flocking tasks are achieved. 

![Fig. 1: \( \phi_i(q) \) plot over different states of \( l_j - l_i \), where above zero is attraction force and below is repulsive (here \( r_s = 11, d = 6 \)).](image)

To accomplished all three Reynolds rules, an additional element for reaching velocity consensus shall be added. Let the adjacency matrix \( A \) with a corresponding \( a_{ij} \) element be

\[
a_{ij} = \rho_n(q_{ji}/r_\alpha),
\]

then the alignment (velocity matching) behavior achieves by applying

\[
u_i^p = a_{ij}(p_j - p_i).
\]

Now we can construct the complete protocol applied on an agent \( i \). Let \( n_{ij} = \sigma_i(q_i - q_j) \) be a vector along the line connecting \( q_i \) to \( q_j \), then for each agent \( i \)

\[
u_i = \sum_{j \in N_i} \phi_i(q)n_{ij} + a_{ij}(p_j - p_i).
\]

The protocol in (18) provides group objective with flocking behavior which avoids the use of a complex, noisy gradient estimation process.

**B. Stability Analysis**

Before proposing the theorem we lay some important assumptions.

**Assumption 2**: Initial distribution conditions:

i) The graph is connected, i.e., \( \max d_G \leq (N-1)r_s \) where \( d_G \) is graph diameter, i.e., each agent has at least one neighbor.

ii) At least one agent is inside the field, i.e. \( \exists i \in N, l_i(q_i(0)) > 0 \).

iii) The agents initial locations at \( t = 0 \) are non-coplanar for 3-D case, i.e.

\[
\exists q_i(0), q_j(0)^T, [q_i(0)^T \times q_j(0)^T]^T \neq 0, \forall j, k \neq i \in N
\]

and are non-collinear for 2-D case, i.e.

\[
\exists q_i(0), \det([q_i(0), q_j(0), q_k(0)^T, 1]) \neq 0, \forall j, k \neq i \in N.
\]

iv) Assume the initial structural energy of the particle system is less than \( c \).

The following lemma introduces the relation between a collective potential function and our proposed protocol.

**Lemma 1**: Under the assumptions above, \( \exists V : \mathbb{R}^{2n} \rightarrow \mathbb{R} \) with certain properties \( V_{ij}(q_j - q_i) \) s.t.

\[
u_{ij}^q = \frac{\partial V_{ij}}{\partial q_{ij}}, \text{ i.e } u^q = -\nabla V.
\]

A sketch of the proof goes as follows. Let us choose a candidate \( V \).

\[
V = \frac{1}{2} \sum \sum \Phi_i(q)
\]

\[
\Phi_i(q) = \int_{d_{ij}}^{q_{ij}} \phi_i(s)ds = \int_{d_{ij} - l_{ij}(q)}^{q_{ij}} \phi_i(s)ds.
\]

Now we will show that \( u = -\nabla V \).

\[
\nabla V = \frac{1}{2} \sum \sum \nabla \Phi_i(q)
\]

\[
= \frac{1}{2} \sum \sum \nabla \int_{d_{ij} - l_{ij}(q)}^{q_{ij}} \phi_i(s)ds
\]

\[
= - \frac{1}{2} \sum \sum n_{ij}(q)\phi_i(q) + \int_{d_{ij} - l_{ij}(q)}^{q_{ij}} \rho_n(q_{ji}/r_\alpha)\phi_s' \nabla l_{ij},
\]

and we receive the first term in (19) for a single agent. The second term is relatively small where \( \rho_n\phi_s' > 0 \) and that \( ||\nabla l_{ij}|| << 1 \).

In a vector form (2) can be written as

\[
\dot{q} = p
\]

\[
\dot{p} = -\nabla V(q) - \dot{L}(q)p,
\]

where \( \dot{L} = L \otimes I_n \). Now, the following lemma provides a characterization of local minima for the collective potential.

**Lemma 2**: All stationary points of \( V(q) \) correspond to \( \alpha \)-Lattice located at the source \( q_i \).

A sketch of the proof goes as follows. First we proof by contradiction that \( \nabla V = 0 \) only when luciferin consensus achieves (i.e. \( l_j = l_i, \forall j \in N_i \)). From Lemma 1, \( u = -\nabla V \), assume that \( l_j \neq l_i \), then with respect to agent \( i \)

\[
\nabla V_{ij} = \rho_n(q_{ji}/r_\alpha)\phi_s(q_{ji} - d_\alpha + l_{ij}) \equiv 0,
\]

iff

\[
q_{ji} - d_\alpha + l_{ij} = 0
\]

\[
q_{ji} = d_\alpha - l_{ij},
\]

but the same is applied with respect to agent \( j \)

\[
q_{ij} = d_\alpha - l_{ij} = d_\alpha + l_{ij},
\]

Since \( q_{ji}, d_\alpha > 0 \) we receive that both conditions hold when \( l_{ij} = -l_{ji} \). Hence, this is only true when \( l_j = l_i \) which contradicts the initial assumption. We also need to show that luciferin consensus with flocking yields convergence to stable equilibrium located at the field maximum, i.e \( q_s \in C \).
By contradiction, \( q_s \notin C \) but still \( l_i = l_j \), \(||q_j - q_i|| = d \), so we look at the case where \( q_i \leq q_j \leq q_s \)

\[
q_j = q_i + d \\
q_i < q_j \\
\psi(q_i) < \psi(q_j) \leq \psi(q_s) \\
\Rightarrow l_i < l_j ,
\]

which contradicts the initial assumption. This also can be done for \( q_j \leq q_i \leq q_s \).

Figure 2 depicts a contour of \( V \) with respect to two agents \( q_1, q_2 \). Only two minimas (marked in *) are received in the location where \(||q_1 - q_2|| = d \) and \( q_s \in C \). Hence, for any initial conditions \( q_1(0), q_2(0) \), the evolution of the dynamics converge to an \( \alpha \)-Lattice configuration located at the source.

Remark - note that \( V(q) \) has two minimas because \( q_1, q_2 \) are replaceable.

Fig. 2: Collective potential function \( V(q) \) contour plot with respect to two agents \( q_1, q_2 \). The symmetric minimas are marked with *.

As suggested in [17] we use moving frame for stability analysis (establish a bounded solutions). Let \( [q_c, p_c] = \text{Avg}([q, c]) \) be the location and velocity center of mass of the frame (flock) respectively,

\[
x_i = q_i - q_c \\
v_i = p_i - p_c ,
\]

so the relatives \( x_j - x_i = q_j - q_i, v_j - v_i = p_j - p_i \) remain the same which applied \( V(q) = V(x) \). Then, (20) can be structure as

\[
\dot{x} = v \\
\dot{v} = - \nabla V(x) - \tilde{L}(x)v
\]

with the corresponding Hamiltonian

\[
H(x, v) = V(x) + K(v). \tag{23}
\]

Notice that the Hamiltonian is constructed from the potential \( V(x) \) and \( K(v) = \frac{1}{2} \sum_{i=2}^{N} ||v_i||^2 \) is a dissipative kinetic energy. In this case, isolated local minima of \( V(x) \) corresponds to the stable equilibrium point. Let \( \Omega_c = \{(x, v) : H(x, v) \leq c \} \) be an invariant set of (22) i.e. all trajectories starting in \( \Omega_c \) remain in \( \Omega_c \).

**Proposition 1:** Consider a group of \( N \) agents with dynamics (22) applying the control law (18). Then, under the described assumptions, any solution starting in \( \Omega_c \), converges to an \( \alpha \)-Lattice configuration located at the source \( q_s \).

**Proof:** Similar to [7], derivation of (23) yields

\[
\dot{H}(x, v) = - v^T \tilde{L}(x)v \\
= - \frac{1}{2} \sum_{(i,j) \in E(x)} a_{ij}(x)||v_j - v_i||^2 \leq 0 , \tag{24}
\]

i.e., the structural energy \( H(x, v) \leq c \) is monotonically decreasing, which guarantees upper bounded velocity,

\[
K(v(t)) \leq H(x(t), v(t)) \leq c .
\]

This also define the speed of convergence with respect to the second smallest eigenvalue of the Laplacian matrix \( \lambda_2(L) \). In addition, under the initial assumptions and Lemma 2, for any solution starting in \( \Omega_c \), there exists an \( R > 0 \) such that \(||x(t)|| \leq R, \forall t \geq 0 \). Combination of the two implies that from any initial location agents inside \( R \), all solutions of (22) are bounded by

\[
||x(t), v(t)||^2 := ||x(t)||^2 + ||v(t)||^2 \leq R^2 + 2c .
\]

Thus, from LaSalle’s invariance principle \( (H(x, v) > 0, \dot{H}(x, v) \leq 0, H(0) = 0) \), every solution asymptotically converges to an equilibrium point \( x^* \), minima of \( V(x) \) and from Lemma 2, this equilibrium corresponds to an \( \alpha \)-Lattice located at the source. In addition, based on (24) and that \( \dot{H} = 0 \) implies \( v = 0 \) we receive that all velocities asymptotically match in the reference frame and consensus is achieved.

**V. SIMULATION AND EXPERIMENTAL RESULTS**

In this section we present both simulation and experimental results. All simulations are conducted in a 3-D space \((m=3)\) where the source is a sphere with the highest concentration located at its center and the agents modeled with double integrator dynamics. Figure 3 captures the convergence and tracking process of the scalar field by 10 agents. The agents’ initial random locations are presented in (a) where not all agents are inside the field. (b) presents final convergence to both flocking structure and extremum location (the connecting lines represent the communication topology), where

\[
d_s = ||q_s - q_c|| < 1e^{-4} \\
||q_j - q_i|| < 1e^{-6}, \forall i, j \in N
\]

so that \( q_s \in C \) and that luciferin level of all agents at equilibrium reaches a steady state which is observed in Figure 4. The tracking ability of the swarm from time-varying changes of the field is captured in (c). Although the filed is moving, the swarm keeps tracking its maxima while maintaining a flocking behavior. The right colored bar, corresponds to agent’s color, indicates the luciferin level. One can notice that initially (a), the agents have different colors, i.e. different luciferin level. Once converging around the source and monitor it, all agents have similar color where luciferin steady-state value is achieved.
Fig. 3: The swarm exploration behavior of a sphere field: (a) agents’ initial location (b) reaching consensus on source location (c) tracking field’s trajectory (in arrow direction) while forming flocking. See animated example: https://www.tuhh.de/t3resources/ics/Upload/fgo.mp4

For evaluating our algorithm performances, we compare between our proposed technique and a straight-forward approach by adding the flocking and GSO elements,

\[
\begin{align*}
    u_i &= \sum_{j \in N_i} \rho_h(q_{ji}/r_{ai})\phi_s(q_{ji} - d_\alpha) + a_\alpha(p_j - p_i) \\
    u_l &= \frac{q_j - q_i}{||q_j - q_i||}, \quad j = \arg \max_{j \in N_i} l_j > l_i \\
    u_i &= k_\alpha u_\alpha + k_l u_l
\end{align*}
\]

where \(k_\alpha, k_l > 0\) are tuning parameters and \(j\) corresponds to the most attractive agent as presented in [5]. The GSO is only applied if exists \(j \in N_i\) s.t. \(l_j > l_i\) otherwise \(u_l = 0\), therefore discontinuities occur. Table I compares the steady-state error \(d_s\) between the two techniques.

<table>
<thead>
<tr>
<th>(N)</th>
<th>Flocking+GSO</th>
<th>FGO</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.014</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0.013</td>
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</tr>
<tr>
<td>17</td>
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</tr>
<tr>
<td>50</td>
<td>1.5</td>
<td>0.6</td>
</tr>
</tbody>
</table>

TABLE I: Comparison between using Flocking with navigational function based on original GSO (25) and the proposed FGO

One can see that the FGO error distance is significantly smaller with comparison to the straight-forward approach. Due to the elimination of discontinuities and the implementation of both flocking and GSO in one potential element. The smooth behavior of FGO comparing to the perturbed output of the straight-forward approach is shown in Figure 5 (a) (one can also notice that FGO converge faster). A steady-state error example of the straight-forward technique is shown in (b) where a single attractive agent is located near the center and all others connected to it. This causes a steady-state distance error relatively larger than the FGO.

An experimental results of \(N = 8\), wheeled robots is presented in Figure 6. The experiment was conducted under the Robotarium project by Georgia Institute of Technology [18]. The 2-D scalar field is the dashed circles with a maxima at its center. First, the agents locate the source and construct flocking structure, then, unknown obstacles (blue-squares) appear and the field starts moving in the positive \(x\) direction. The agents keep tracking the field while avoiding obstacles during the whole movement and until the field comes to a rest. The obstacle avoidance technique is beyond the scope of this paper. The general idea is to use a distributed algorithm estimating obstacle center and radius for generating a virtual repulsive force around it.

VI. CONCLUSIONS

In this paper we presented FGO - Flocking Glowworm Optimization, a technique which provides a solution for tracking time-varying scalar field. Motivated to avoid gradient estimation methods or formation control design, we use the framework of flocking with a combination of glowworm optimization protocol. Here, as opposed to using a
Future work will focus on more realistic scenarios such as multi extrema fields and different agents models.

REFERENCES