VECTOR SPACE BASES FOR THE HOMOGENEOUS PARTS IN HOMOGENEOUS IDEALS AND GRADED MODULES OVER A POLYNOMIAL RING

Natalia Dück\textsuperscript{1}§, Karl-Heinz Zimmermann\textsuperscript{2}

\textsuperscript{1,2}Hamburg University of Technology
21071, Hamburg, GERMANY

Abstract: In this paper, vector space bases for the homogeneous parts of homogeneous ideals and graded modules over a commutative polynomial ring are given using Gröbner bases.

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1. Introduction

Gröbner bases have originally been introduced by Buchberger for the algorithmic solution of some fundamental problems in commutative algebra [3]. Since then Gröbner bases evolved into a crucial concept in symbolic computations providing a uniform approach to solving a wide range of problems such as effective computations in residue class rings modulo polynomial ideals and in modules over polynomial rings, and calculating syzygies and graded resolutions for homogeneous ideals [2, 4, 6].

In this paper, we provide bases for the vector spaces corresponding to the homogeneous parts of homogeneous ideals or graded modules over a polynomial ring. Moreover, we show that the vector space basis for the homogenous part
of a homogeneous ideal is also a Gröbner basis for the ideal generated by the homogeneous part if the degree of the homogeneous part is large enough. The required notions and definitions are introduced in Section 2 and the results are provided in the Sections 3 and 4.

2. Graded Rings and Gröbner Bases for Modules

Let $R = \mathbb{K}[x_0, x_1, \ldots, x_n]$ denote the commutative polynomial ring in $n + 1$ indeterminates over a field $\mathbb{K}$. The monomials in $R$ are denoted by $x^u = x_0^{u_0} \cdots x_n^{u_n}$, where $u = (u_0, \ldots, u_n) \in \mathbb{N}_0^{n+1}$. The total degree of a monomial $x^u$ is the sum $|u| = u_1 + \ldots + u_n$. The ring $R$ has a natural grading in the sense that it admits a direct sum decomposition

$$R = \bigoplus_{s \geq 0} R_s,$$

where for each integer $s \geq 0$ the set $R_s$ is the additive subgroup of $R$ that consists of all homogeneous polynomials of degree $s$ plus the zero polynomial, and the complex product $R_sR_t = \{rr' \mid r \in R_s, r' \in R_t\}$ is contained in $R_{s+t}$ for all $s, t \geq 0$. Note that $R_0 = \mathbb{K}$ and $R_0R_s \subseteq R_s$. Thus the subgroups $R_s$ are also $\mathbb{K}$-vector spaces.

A module $M$ over $R$ (or any graded ring) is a graded module over $R$ if it can be decomposed as

$$M = \bigoplus_{t \in \mathbb{Z}} M_t,$$

where each $M_t$ is an additive subgroup of the additive group of $M$ with the property that the complex product $R_sM_t = \{rm \mid r \in R_s, m \in M_t\}$ lies in $M_{s+t}$ for all $s \geq 0$ and all $t \in \mathbb{Z}$. Each additive subgroup $M_t$ is a module over $R_0 = \mathbb{K}$ since $R_0M_t \subseteq M_t$. Thus the subgroups $M_t$ are also $\mathbb{K}$-vector spaces.

Let $m \geq 1$ be an integer. The free $R$-module $R^m$ has the standard basis consisting of the canonical unit vectors $e_1, \ldots, e_m$. The module $R^m$ is graded over $R$ with the (standard) grading

$$(R^m)_t = (R_t)^m, \quad t \in \mathbb{Z}.$$  

Note that $(R^m)_t = \{0\}$ if $t \leq 0$.

A monomial in $R^m$ is an element of the form $x^ue_i$ for some $1 \leq i \leq m$ and $u \in \mathbb{N}_0^{n+1}$. Each element in $R^m$ can be uniquely written as a $\mathbb{K}$-linear
combination of monomials. For instance, let $R = \mathbb{K}[x, y]$ and take the following element in $R^2$:

$$
\begin{pmatrix}
3x^2y + xy^2 + 1 \\
x^3y^2 + 2xy^5 - 3y
\end{pmatrix} = 3x^2ye_1 + xy^2e_1 + e_1 + x^3y^2e_2 + 2xy^5e_2 - 3ye_2.
$$

A monomial order on $R^m$ is a relation $\succ$ on the set of monomials in $R^m$ satisfying the following conditions: (1) $\succ$ is a total order, (2) $\succ$ is well-ordering, and (3) for any monomials $m, m' \in R^m$, $m \succ m'$ implies $x^u m \succ x^u m'$ for each monomial $x^u \in R$.

Any monomial order on $R$ can be extended to a monomial order on $R^m$. For this, an ordering of the standard basis vectors needs to fixed, say by downward ordering $e_1 > \cdots > e_m$. Then the TOP (term over position) extension of a monomial order $\succ$ on $R$, denoted by $\succ_{TOP}$, is defined as

$$
x^ue_i \succ_{TOP} x^ve_j : \iff x^u \succ x^v \vee (x^u = x^v \land i < j)
$$

and the POT (position over term) extension of $\succ$, denoted by $\succ_{POT}$, is given by

$$
x^ue_i \succ_{POT} x^ve_j : \iff i < j \vee (i = j \land x^u \succ x^v).
$$

For instance, if the lex order on $R = \mathbb{K}[x, y]$ with $x > y$ is extended to a TOP order on $R^2$, then

$$
x^3y^2e_2 \succ_{TOP} 3x^2ye_1 \succ_{TOP} 2xy^5e_2 \succ_{TOP} xy^2e_1 \succ_{TOP} -3ye_2 \succ_{TOP} e_1.
$$

However, if the lex order on $R$ is extended to a POT order on $R^2$, then

$$
3x^2ye_1 \succ_{POT} -2xy^2e_1 \succ_{POT} e_1 \succ_{POT} x^3y^2e_2 \succ_{POT} 2xy^5e_2 \succ_{POT} -3ye_2.
$$

Given a monomial order $\succ$ on $R^m$, each non-zero polynomial $f \in R^m$ has a unique leading term given by the largest involved term and denoted by $\text{lt}_{\succ}(f)$; the corresponding leading monomial is referred to as $\text{lm}_{\succ}(f)$. Each submodule $M$ of $R^m$ has a leading submodule generated as a module by the leading terms of its elements,

$$
\langle \text{lt}_{\succ}(M) \rangle = \langle \{\text{lt}_{\succ}(f) \mid f \in M\} \rangle.
$$

A Gröbner basis for a submodule $M$ of $R^m$ w.r.t. a monomial order $\succ$ on $R^m$ is a finite subset $G$ of $M$ with the property that the leading terms of the elements in $G$ generate the leading submodule of $M$, i.e.,

$$
\langle \text{lt}_{\succ}(M) \rangle = \langle \{\text{lt}_{\succ}(g) \mid g \in G\} \rangle.
$$
Each submodule of $R^m$ has a Gröbner basis which is generally not uniquely determined. However, a unique Gröbner basis $\mathcal{G}$ called reduced Gröbner basis can be obtained, where the leading terms of the elements in $\mathcal{G}$ are monic and for two distinct elements $g$ and $g'$ in $\mathcal{G}$ no term involved in $g$ is divisible by the leading term of $g'$. Gröbner bases can be computed by Buchberger’s algorithm for submodules which is available by almost every computer algebra system. More details on Gröbner bases and modules can be found in [1, 5].

3. Vector Space Bases for the Homogeneous Parts in Homogeneous Ideals

An ideal $I$ in $R$ is homogeneous if for any element $f \in I$ the homogeneous components of $f$ are also in $I$. A homogeneous ideal $I$ in $R$ is a graded submodule of $R$ with the direct sum decomposition

$$I = \bigoplus_{t \in \mathbb{Z}} I_t,$$

where the homogeneous parts are given by $I_t = I \cap R_t$ for all $t \in \mathbb{Z}$. Note that $I_t = \{0\}$ if $t \leq 0$.

A $\mathbb{K}$-basis for the homogeneous part $R_t$ is given by all monomials of total degree $t$ and so we have

$$\dim_{\mathbb{K}} R_t = \binom{t + n - 1}{n - 1}, \quad t \in \mathbb{N}_0.$$

Thus the homogeneous part $I_t$ is a finite-dimensional vector space for each $t \in \mathbb{Z}$.

The quotient module $R/I$ has also a graded module structure defined by

$$(R/I)_t = R_t/I_t = R_t/(I \cap R_t), \quad t \in \mathbb{Z}.$$

By the dimension formula,

$$\dim_{\mathbb{K}} R_t = \dim_{\mathbb{K}} I_t + \dim_{\mathbb{K}} (R/I)_t, \quad t \in \mathbb{N}_0,$$

and thus the quotient spaces $(R/I)_t$ are also finite dimensional. Moreover, the ideal of leading terms of $I$ fulfills

$$\dim_{\mathbb{K}} R_t/I_t = \dim_{\mathbb{K}} R_t/\langle \text{lt}_{>}(I) \rangle_t, \quad t \in \mathbb{N}_0,$$

where $\langle \text{lt}_{>}(I) \rangle_t = \langle \text{lt}_{>}(I) \rangle \cap R_t$.

Note that the additive subgroups $I_t$ are not ideals. Nonetheless, we can consider the ideal $\langle I_t \rangle$ generated by the elements in $I_t$. 
Proposition 1. Let I be a homogeneous ideal in R. Let \( G \) be the reduced Gröbner basis for I w.r.t. any monomial order \( \succ \) on R and let

\[
\text{lm}_\succ(I_t) = \{ \text{lm}_\succ(f) \mid f \in I_t \} = \{ x^{a_1}, \ldots, x^{a_s} \}, \quad t \in \mathbb{N}_0.
\]

Then a \( \mathbb{K} \)-basis for the vector space \( I_t \) is given by the set of binomials

\[
B_t = \{ x^{a_1} - r_1, \ldots, x^{a_s} - r_s \},
\]

where \( r_i \) is the remainder of \( x^{a_i} \) on division by \( G \) for \( 1 \leq i \leq s \).

The vector space \( I_t \) is non-trivial if and only if \( t \geq \min \{ \deg(\text{lt}_\succ(g)) \mid g \in G \} \). If \( t \geq \max \{ \deg(\text{lt}_\succ(g)) \mid g \in G \} \), then the set \( \text{lm}_\succ(I_t) \) consists of all monomial multiples of the elements in \( \{ \text{lt}_\succ(g) \mid g \in G \} \) which are of total degree \( t \), and \( B_t \) is the reduced Gröbner basis for the homogeneous ideal \( \langle I_t \rangle \).

Proof. Note that the reduced Gröbner basis \( G \) for a homogeneous ideal always consists of homogeneous polynomials and the remainder of a homogeneous polynomial \( f \) divided by a set of homogeneous polynomials is either zero or homogeneous of the same total degree as \( f \). It follows that if a monomial \( x^{a_i} \) of total degree \( t \) is divided by the Gröbner basis \( G \) giving the remainder \( r_i \), then the difference \( x^{a_i} - r_i \) will be a polynomial of total degree \( t \) with leading term \( x^{a_i} \) which lies in \( I_t \), \( 1 \leq i \leq s \). Hence, \( B_t \) is contained in \( I_t \).

By rearranging and deleting duplicates, we may assume that \( x^{a_1} \succ \ldots \succ x^{a_s} \). Let \( f_i = x^{a_i} - r_i \in I_t \), \( 1 \leq i \leq s \), and claim that the elements \( f_1, \ldots, f_s \) form a \( \mathbb{K} \)-basis of \( I_t \). Indeed, consider a nontrivial linear combination \( k_1 f_1 + \ldots + k_t f_t \) with \( k_i \in \mathbb{K} \) and take the smallest index \( i \) such that \( k_i \neq 0 \). By the ordering of the leading monomials, there is nothing to cancel \( k_i f_i \) and so the linear combination is nonzero. Hence, \( f_1, \ldots, f_s \) are linearly independent.

Moreover, let \( U \) be the subspace of \( I_t \) spanned by \( f_1, \ldots, f_s \). Suppose \( U \) is a proper subspace of \( I_t \). Pick an element \( f \in I_t \setminus U \) whose leading monomial is minimal. By definition, the leading monomial of \( f \) equals the leading monomial of \( f_i \) for some \( i \) and \( \text{lt}(f) = k \text{lt}(f_i) \) for some \( k \in \mathbb{K} \). It follows that \( f - k f_i \) lies in \( I_t \) and has a smaller leading monomial. Thus \( f - k f_i \in U \) by the minimality of the leading monomial of \( f \) and so \( f \in U \), a contradiction. Hence, \( U = I_t \) and the claim follows.

Let \( t \geq \min \{ \deg(\text{lt}_\succ(g)) \mid g \in G \} \). Then \( t \geq \deg(g) \) for some element \( g \in G \) and thus the leading monomial of \( gx^u \) with \( |u| + \deg(g) = t \) lies in \( \text{lm}_\succ(I_t) \). Hence, the vector space \( I_t \) is non-trivial. Conversely, let \( f \in I_t \). Then \( f \in I \) and there is an element \( g \in G \) such that the leading term of \( f \) is divisible by the leading term of \( g \). Hence, \( t \geq \deg(g) \).
Finally, let \( t \geq \max\{\deg(\text{lt}_\succ(g)) \mid g \in G\} \) and claim that \( B_t \) is the reduced Gröbner basis for \( \langle I_t \rangle \). Indeed, the set \( B_t \) generates \( I_t \) as a vector space and so generates also the ideal \( \langle I_t \rangle \). It remains to show that the leading terms of the elements in \( B_t \) generate the leading ideal \( \langle \text{lt}_\succ(\langle I_t \rangle) \rangle \). To this end, let \( f \in \langle I_t \rangle \). Since \( f \in I_t \), there is a Gröbner basis element \( g \in G \) such that \( \text{lt}_\succ(\langle g \rangle) \) divides \( \text{lt}_\succ(f) \). By the choice of \( t \), \( \deg(g) \leq t \) and so all monomial multiples of \( \text{lt}_\succ(g) \) of total degree \( t \) appear as leading terms in \( B_t \). But the leading term of \( f \) is also a monomial multiple of \( \text{lt}_\succ(g) \) (possibly of total degree larger than \( t \)) and so must be divisible by at least one monomial \( x^{a_i} \) where \( 1 \leq i \leq s \). This proves the claim.

The set \( \text{lm}_\succ(I_t) \) can be constructed from the reduced Gröbner basis \( G \) for \( I \) w.r.t. any monomial order \( \succ \) as follows. Starting with the empty set add for each element \( g \in G \) with leading term of total degree \( s \leq t \) all monomial multiples of \( \text{lt}_\succ(g) \) that are of degree \( t \), i.e., the set \( \{\text{lt}_\succ(g)x^u \mid |u| = t-s\} \). This will give the set \( \text{lm}_\succ(I_t) \) in a finite number of steps, since the Gröbner basis is finite.

**Example 1.** Consider the homogeneous ideal

\[
I = \langle z^3 - yw^2, yz - xw, y^3 - x^2z, xz^2 - y^2w \rangle \subset \mathbb{K}[x, y, z, w] = R.
\]

The above set is the reduced Gröbner basis w.r.t. the grevlex order \( \succ \) on \( R \) with \( x \succ y \succ z \succ w \). Thus

\[
\langle \text{lt}_\succ(I) \rangle = \langle z^3, yz, y^3, xz^2 \rangle.
\]

Note that all monomials in this generating set have degree greater than or equal to 2 and so \( I_i = \{0\} \) for \( i \leq 1 \). By the above remark, the leading ideals for \( I_2 \), \( I_3 \) and \( I_4 \) are generated as follows,

\[
\begin{align*}
\text{lm}_\succ(I_2) &= \{yz\}, \\
\text{lm}_\succ(I_3) &= \{z^3, xyz, y^2z, yzw, y^3, xz^2\}, \\
\text{lm}_\succ(I_4) &= \{xz^3, yz^3, z^4, wz^3, x^2yz, y^3z, yzw^2, xy^2z, y^2z^2, yz^2w, \\
&\quad xz^2, y^2zw, xz^2, y^3, y^3w, x^2z^2, x^2w\}.
\end{align*}
\]

By the division algorithm, a vector space basis for \( I_2 \) is \( B_2 = \{yz - xw\} \), a vector space basis for \( I_3 \) is

\[
B_3 = \{z^3 - yw^2, xyz - x^2w, y^2z - xyw, yz^2 - xzw, \\
yzw - xw^2, y^3 - x^2z, xz^2 - y^2w\}.
\]
and a vector space basis for $I_4$ is

$$B_4 = \left\{ \begin{array}{llll}
xz^3 - xyw^2, & yz^3 - y^2w^2, & z^4 - xw^3, \\
z^3w - yw^3, & x^2yz - x^3w, & y^3z - xy^2w, \\
yzw^2 - xw^3, & xy^2z - x^2yw, & y^2z^2 - x^2w^2, \\
yz^2w - xzw^2, & xyz^2 - x^2zw, & yzw^2 - xyw^2, \\
xyzw - x^2w^2, & xy^3 - x^3z, & y^4 - x^3w, \\
y^3w - x^2zw, & x^2z^2 - xy^2w, & xzw^2 - y^2w^2 \end{array} \right\}.$$  

The homogeneous ideal $\langle I_3 \rangle$ has the generating set $B_3$ and one can show that it is also the reduced Gröbner basis for this ideal w.r.t. the above grevlex order. However, a Gröbner basis for the ideal $I$ w.r.t. the grevlex order with $w \succ y \succ z \succ x$ is

$$\{z^4 - xw^3, yw^2 - z^3, yz - xw, y^2w - xz^2, y^3 - x^2z\}.$$

It follows that a basis for the $\mathbb{K}$-vector space $I_3$ is

$$B_3' = \{yw^2 - z^3, xyz - x^2w, y^2z - xyw, yzw - xzw, \\
yzw - xw^2, y^2w - xz^2, y^3 - x^2z\}.$$  

Since $B_3'$ differs from $B_3'$ only by scalar multiples, it is also a generating set for the ideal $\langle I_3 \rangle$. However, it is not the reduced Gröbner basis w.r.t. the grevlex order with $w \succ y \succ z \succ x$ since the S-polynomial

$$S(yw^2 - z^3, yzw - xw^2) = z(yw^2 - z^3) - w(yzw - xw^2) = -z^4 + xw^3$$

has the leading term $z^4$ which is not divisible by any of the leading terms in $B_3'$. This confirms the necessity of the condition $t \geq \max\{\deg(lt_>(g)) \mid g \in \mathcal{G}\}$ for $B_t$ to form a Gröbner basis for the ideal $\langle I_t \rangle$.

4. Vector Space Bases for the Homogeneous Parts in Graded Modules

Let $m \geq 1$ be an integer. The graded submodules $M$ of $R^m$ can be characterized as follows [5]:

- The standard grading on $R^m$ induces a graded module structure on $M$, which is given by $M_t = (R^m)_t \cap M$ for all $t \in \mathbb{Z}$. 

• There are elements $f_1, \ldots, f_r$ in $R^m$, whose components are homogeneous polynomials of the same degree, such that $M = \langle f_1, f_2, \ldots, f_r \rangle \subset R^m$ for all $t \in \mathbb{Z}$.

• A reduced Gröbner basis for $M$ (w.r.t. any monomial order on $R^m$) consists of vectors of homogeneous polynomials whose components have the same degree.

Using these facts, we obtain the following result.

**Proposition 2.** Let $M \subset R^m$ be a graded module over $R$. Let $G$ be a Gröbner basis for $M$ w.r.t. any monomial order $\succ$ and let

$$\text{lm}_\succ(M_t) = \{\text{lm}_\succ(f) \mid f \in M_t\} = \{x^{a_1} e_{i_1}, \ldots, x^{a_s} e_{i_s}\}, \quad t \in \mathbb{N}_0,$$

where $e_{i_1}, \ldots, e_{i_s}$ are unit vectors in $R^m$. Then a $\mathbb{K}$-basis for the vector space $M_t$ is given by

$$B_t = \{x^{a_1} e_{i_1} - r_1, \ldots, x^{a_s} e_{i_s} - r_s\},$$

where $r_j \in R^m$ is the remainder of $x^{a_j} e_{i_j}$ on division by $G$ for each $1 \leq j \leq s$. The vector space $M_t$ is non-trivial if and only if $t \geq \min\{\deg(\text{lt}_\succ(g)) \mid g \in G\}$.

The proof is the same as that of Prop. 1 since all statements used there are also applicable to submodules of $R^m$ (see for instance [5, Chapter 5, 2]). Moreover, the construction of $\text{lm}_\succ(M_t)$ in the module case is analogous to that in the ideal case (see the remark after the proof of Prop. 1).

**Example 2.** Let $R = \mathbb{K}[x, y, z, w]$ and consider the submodule $M$ of $R^4$ generated by the vectors

$$\begin{pmatrix} y^2 \\ xz \\ yw \\ z^2 \end{pmatrix}, \quad \begin{pmatrix} z \\ w \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ -z \\ -w \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ x \end{pmatrix}.$$

The generators are vectors of homogeneous monomials of the same total degree and so by the above remark, the module $M$ is graded.

The reduced Gröbner basis for the module $M$ w.r.t. the POT-extension of the grevlex order $\succ$ with $x \succ y \succ z \succ w$ is given by the columns of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & z & x & y^2 \\ 0 & yz - wx & xz^2 - y^2w & y^3 - x^2z & w & yxz \\ x & -z^2 & yzw & -y^2z & 0 & -z & yw \\ y & -zw & z^3 & -xz^2 & 0 & -w & z^2 \end{pmatrix}.$$
Thus the leading ideal of $M$ is
\[ \langle \text{lt}_{POT}(M) \rangle = \langle y^2 e_1, xe_1, ze_1, y^3 e_2, xz^2 e_2, yze_2, xe_3 \rangle \]
and therefore
\[ \text{lm}_{POT}(M_1) = \{ xe_1, ze_1, xe_3 \}, \]
\[ \text{lm}_{POT}(M_2) = \{ x^2 e_1, xye_1, xze_1, xwe_1, y^2 e_1, yze_1, z^2 e_1, \\
zw e_1, yze_2, x^2 e_3, xye_3, xze_3, xwe_3 \}. \]

The bases for the $\mathbb{K}$-vector spaces $B_1$ and $B_2$ for $M_1$ and $M_2$, respectively, are
\[
B_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ x \\ y \\ z \end{pmatrix}, \begin{pmatrix} w \\ 0 \\ -z \\ -w \end{pmatrix} \right\},
\]
\[
B_2 = \left\{ \begin{pmatrix} x^2 \\ xy \\ -xz \\ -xw \end{pmatrix}, \begin{pmatrix} y^2 \\ y^2 \\ -yz \\ -yw \end{pmatrix}, \begin{pmatrix} xz \\ xw \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} yw \\ zw \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} yz \\ zw \\ 0 \\ 0 \end{pmatrix} \right\}.
\]

References


