Linear free vibrations with uncertain initial conditions

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Abstract
We study a one degree of freedom, linear free vibrating mass system where there is uncertainty about how the system starts off. The uncertain initial conditions are modeled by fuzzy numbers and an analytical approach is used to propagate the uncertainties through the computations. As a result, we obtain closed-form symbolic expressions for the whole spectrum of possible calculation results.

1 Introduction

There is an increasing effort in the scientific community to provide suitable methods for the inclusion of uncertainties into mathematical models. One way to do so is to introduce parametric uncertainty by representing the uncertain model parameters as fuzzy numbers [3] and evaluating the model equations by using ZADEH’s extension principle [8].

Recently, an analytical approach to evaluating monotonic functions of fuzzy numbers was introduced by the authors [6], which is based on an alternative formulation of the extension principle [2]. In this paper, we use this approach to study a one degree of freedom, linear free vibrating mass system with fuzzy initial conditions. As a result, we obtain closed-form symbolic expressions for the whole spectrum of possible calculation results.

2 Fuzzy numbers

Fuzzy numbers [3] are a special class of fuzzy sets [7], which can be defined as follows.

A normal, convex fuzzy set \( \tilde{x} \) over the real line \( \mathbb{R} \) is called fuzzy number if there is exactly one \( \bar{x} \in \mathbb{R} \) with \( \mu_{\tilde{x}}(\bar{x}) = 1 \) and the membership function is at least piecewise continuous. The value \( \bar{x} \) is called the modal or peak value of \( \tilde{x} \).

Theoretically, an infinite number of possible types of fuzzy numbers can be defined. However, only few of them are important for engineering applications [5]. These typical fuzzy numbers shall be described in the following.

2.1 Triangular fuzzy numbers

Due to its very simple, linear membership function, the triangular fuzzy number (TFN) is the most frequently used fuzzy number in engineering. In order to define a TFN with the membership function

\[
\mu_{\tilde{x}}(x) = \begin{cases} 
1 + \frac{x - \bar{x}}{\tau_L}, & \bar{x} - \tau_L \leq x \leq \bar{x}, \\
1 - \frac{x - \bar{x}}{\tau_R}, & \bar{x} < x \leq \bar{x} + \tau_R,
\end{cases}
\]  

(1)
we use the parametric notation [5]
\[ \tilde{x} = \text{tfn}(\bar{x}, \tau^L, \tau^R), \]
where \( \bar{x} \) denotes the modal value, \( \tau^L \) the left-hand, and \( \tau^R \) the right-hand spread of \( \tilde{x} \), cf. Figure 1(a). If \( \tau^L = \tau^R \), the TFN is called symmetric. Its \( \alpha \)-cuts \( x(\alpha) = [x^L(\alpha), x^R(\alpha)] \) result from the inverse functions of Eqs. (1) with respect to \( x \):
\[ x^L(\alpha) = \bar{x} - \tau^L(1 - \alpha), \quad 0 < \alpha \leq 1, \]
\[ x^R(\alpha) = \bar{x} + \tau^R(1 - \alpha), \quad 0 < \alpha \leq 1. \]

2.2 Gaussian fuzzy numbers
Another widely-used fuzzy number in engineering is the Gaussian fuzzy number (GFN), which is based on the normal distribution from probability theory. In order to define a GFN with the membership function
\[ \mu_{\tilde{x}}(x) = \begin{cases} \exp\left[-\frac{1}{2} \left( \frac{x - \bar{x}}{\sigma^L} \right)^2 \right], & x \leq \bar{x}, \\ \exp\left[-\frac{1}{2} \left( \frac{x - \bar{x}}{\sigma^R} \right)^2 \right], & x > \bar{x}, \end{cases} \]
we use the parametric notation [5]
\[ \tilde{x} = \text{gfn}(\bar{x}, \sigma^L, \sigma^R), \]
where \( \bar{x} \) denotes the modal value, \( \sigma^L \) the left-hand, and \( \sigma^R \) the right-hand standard deviation of \( \tilde{x} \), cf. Figure 1(b). If \( \sigma^L = \sigma^R \), the GFN is called symmetric. Its \( \alpha \)-cuts \( x(\alpha) = [x^L(\alpha), x^R(\alpha)] \) result in
\[ x^L(\alpha) = \bar{x} - \sigma^L \sqrt{-2 \ln(\alpha)}, \quad 0 < \alpha \leq 1, \]
\[ x^R(\alpha) = \bar{x} + \sigma^R \sqrt{-2 \ln(\alpha)}, \quad 0 < \alpha \leq 1. \]

3 Analytical approach
Let \( \tilde{x}_1, \ldots, \tilde{x}_n \) be \( n \) independent fuzzy numbers with the membership functions \( \mu_{\tilde{x}_1}(x_1), \ldots, \mu_{\tilde{x}_n}(x_n) \), and let \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) be a continuous function with \( y = f(x_1, \ldots, x_n) \). Furthermore, let \( f \) be (strictly) monotonic increasing in \( x_i \), \( i = 1, \ldots, k \), and (strictly) monotonic decreasing in \( x_j \), \( j = 1, \ldots, \ell \), in the domain of interest, and let \( k + \ell = n \). Then, the \( \alpha \)-cuts \( y(\alpha) = [y^L(\alpha), y^R(\alpha)] \) of \( \tilde{y} \) are determined by [6]
\[ y^L(\alpha) = f(x^L_i(\alpha), x^R_j(\alpha)), \quad 0 < \alpha \leq 1, \]
\[ y^R(\alpha) = f(x^R_i(\alpha), x^L_j(\alpha)), \quad 0 < \alpha \leq 1. \]
4 Linear free vibrations

Consider a linear system with one degree of freedom consisting of a block with mass $m$ moving on a smooth surface as illustrated in Figure 2. The block is connected to a wall with a linear spring with spring constant $k$ and a viscous damper with damping coefficient $d$. This system is governed by the following linear, homogeneous ordinary differential equation [4]:

$$\ddot{x} + 2\xi\dot{x} + \omega^2 x = 0,$$

(3)

where

$$\omega = \sqrt{\frac{k}{m}}$$

denotes the natural frequency and

$$\xi = \frac{d}{2m}$$

the normalized damping coefficient of the system.

The model parameters $m$, $k$, and $d$ are assumed to be known. However, if there is uncertainty about how the system starts off, the initial conditions $x_0$ and $\dot{x}_0$ can be modeled by triangular fuzzy numbers [1].

4.1 No damping

In case of $\xi = 0$, the general solution of Eq. (3) is given by [4]

$$x(t) = x_0 \cos(\omega t) + \dot{x}_0 \frac{\sin(\omega t)}{\omega},$$

where $x_0 = x(0)$ and $\dot{x}_0 = \dot{x}(0)$ denote the initial conditions.

Since for $k \in \mathbb{N}_0$

$$\frac{\partial x(t)}{\partial x_0} = \cos(\omega t) \begin{cases} \geq 0, & \omega t \in [0 + 2k\pi, 0.5\pi + 2k\pi] \cup [1.5\pi + 2k\pi, 2\pi + 2k\pi], \\
< 0, & \omega t \in (0.5\pi + 2k\pi, 1.5\pi + 2k\pi), \end{cases}$$

$$\frac{\partial x(t)}{\partial \dot{x}_0} = \frac{\sin(\omega t)}{\omega} \begin{cases} \geq 0, & \omega t \in [0 + 2k\pi, \pi + 2k\pi], \\
< 0, & \omega t \in (\pi + 2k\pi, 2\pi + 2k\pi), \end{cases}$$

we can distinguish between four domains.
4.1.1 \( \omega t \in [0 + 2k\pi, 0.5\pi + 2k\pi] \)

Here, \( x(t) \) is (strictly) monotonic increasing in both \( x_0 \) and \( \dot{x}_0 \) for positive values. Hence, the \( \alpha \)-cuts \( x(\alpha, t) = [x^L(\alpha, t), x^R(\alpha, t)] \) of \( \ddot{x}(t) \) are

\[
x^L(\alpha, t) = x(x^L_0(\alpha), \dot{x}_0^L(\alpha), t) = (\bar{x}_0 - \tau^L_{x_0}(1 - \alpha)) \cos(\omega t)
\]

\[
+ (\ddot{x}_0 - \tau^L_{\dot{x}_0}(1 - \alpha)) \frac{\sin(\omega t)}{\omega}, \quad 0 < \alpha \leq 1,
\]

\[
x^R(\alpha, t) = x(x^R_0(\alpha), \dot{x}_0^R(\alpha), t) = (\bar{x}_0 + \tau^R_{x_0}(1 - \alpha)) \cos(\omega t)
\]

\[
+ (\ddot{x}_0 + \tau^R_{\dot{x}_0}(1 - \alpha)) \frac{\sin(\omega t)}{\omega}, \quad 0 < \alpha \leq 1.
\]

With

\[
x^L(0, t) = (\bar{x}_0 - \tau^L_{x_0}) \cos(\omega t) + (\ddot{x}_0 - \tau^L_{\dot{x}_0}) \frac{\sin(\omega t)}{\omega}, \quad x^L(0, t) < x(t) \leq x^L(1, t),
\]

\[
x^R(1, t) = (\bar{x}_0 + \tau^R_{x_0}) \cos(\omega t) + (\ddot{x}_0 + \tau^R_{\dot{x}_0}) \frac{\sin(\omega t)}{\omega}, \quad x^R(1, t) < x(t) < x^R(0, t).
\]

### 4.1.2 \( \omega t \in (0.5\pi + 2k\pi, \pi + 2k\pi] \)

Here, \( x(t) \) is (strictly) monotonic decreasing in \( x_0 \) and (strictly) monotonic increasing in \( \dot{x}_0 \) for positive values. Hence, the \( \alpha \)-cuts \( x(\alpha, t) = [x^L(\alpha, t), x^R(\alpha, t)] \) of \( \ddot{x}(t) \) are

\[
x^L(\alpha, t) = x(x^L_0(\alpha), \dot{x}_0^L(\alpha), t) = (\bar{x}_0 + \tau^L_{x_0}(1 - \alpha)) \cos(\omega t)
\]

\[
+ (\ddot{x}_0 - \tau^L_{\dot{x}_0}(1 - \alpha)) \frac{\sin(\omega t)}{\omega}, \quad 0 < \alpha \leq 1,
\]

\[
x^R(\alpha, t) = x(x^R_0(\alpha), \dot{x}_0^R(\alpha), t) = (\bar{x}_0 - \tau^R_{x_0}(1 - \alpha)) \cos(\omega t)
\]

\[
+ (\ddot{x}_0 + \tau^R_{\dot{x}_0}(1 - \alpha)) \frac{\sin(\omega t)}{\omega}, \quad 0 < \alpha \leq 1.
\]

With

\[
x^L(0, t) = (\bar{x}_0 + \tau^R_{x_0}) \cos(\omega t) + (\ddot{x}_0 - \tau^L_{\dot{x}_0}) \frac{\sin(\omega t)}{\omega},
\]

\[
x^L(1, t) = \bar{x}_0 \cos(\omega t) + \ddot{x}_0 \frac{\sin(\omega t)}{\omega} = x^R(1, t),
\]

\[
x^R(0, t) = (\bar{x}_0 - \tau^L_{x_0}) \cos(\omega t) + (\ddot{x}_0 + \tau^R_{\dot{x}_0}) \frac{\sin(\omega t)}{\omega},
\]

the membership function of \( \ddot{x}(t) \) yields

\[
\mu_{\ddot{x}(t)}(x(t)) = \left\{ \begin{array}{ll}
-\omega(\bar{x}_0 + \tau^R_{x_0}) \cos(\omega t) - (\ddot{x}_0 - \tau^L_{\dot{x}_0}) \frac{\sin(\omega t)}{\omega}, & \text{for } x^L(0, t) < x(t) \leq x^L(1, t), \\
-\omega(\bar{x}_0 + \tau^R_{x_0}) \cos(\omega t) + \tau^L_{\dot{x}_0} \sin(\omega t), & x^L(0, t) < x(t) \leq x^R(1, t), \\
+\omega(\bar{x}_0 + \tau^R_{x_0}) \cos(\omega t) + (\ddot{x}_0 + \tau^R_{\dot{x}_0}) \frac{\sin(\omega t)}{\omega} - \omega x(t), & x^R(1, t) < x(t) < x^R(0, t).
\end{array} \right.
\]
4.1.3 $\omega t \in (\pi + 2k\pi, 1.5\pi + 2k\pi)$

Here, $x(t)$ is (strictly) monotonic decreasing in both $x_0$ and $\dot{x}_0$ for positive values. Hence, the $\alpha$-cuts $x(\alpha, t) = [x_L(\alpha, t), x_R(\alpha, t)]$ of $\dot{x}(t)$ are

\[
x^L(\alpha, t) = x(x_0^R(\alpha), \dot{x}_0^R(\alpha), t) = (\bar{x}_0 + \tau_{x_0}^R(1 - \alpha)) \cos(\omega t) \\
+ (\bar{x}_0 + \tau_{x_0}^R(1 - \alpha)) \frac{\sin(\omega t)}{\omega}, \quad 0 < \alpha \leq 1,
\]

\[
x^R(\alpha, t) = x(x_0^L(\alpha), \dot{x}_0^L(\alpha), t) = (\bar{x}_0 - \tau_{x_0}^L(1 - \alpha)) \cos(\omega t) \\
+ (\bar{x}_0 - \tau_{x_0}^L(1 - \alpha)) \frac{\sin(\omega t)}{\omega}, \quad 0 < \alpha \leq 1.
\]

With

\[
x^L(0, t) = (\bar{x}_0 + \tau_{x_0}^R) \cos(\omega t) + (\bar{x}_0 + \tau_{x_0}^R) \frac{\sin(\omega t)}{\omega}, \quad x^L(0, t) < x(t) \leq x^L(1, t),
\]

\[
x^L(1, t) = \bar{x}_0 \cos(\omega t) + \bar{x}_0 \frac{\sin(\omega t)}{\omega} = x^R(1, t),
\]

\[
x^R(0, t) = (\bar{x}_0 - \tau_{x_0}^L) \cos(\omega t) + (\bar{x}_0 - \tau_{x_0}^L) \frac{\sin(\omega t)}{\omega}, \quad x^R(1, t) < x(t) < x^R(0, t).
\]

4.1.4 $\omega t \in (1.5\pi + 2k\pi, 2\pi + 2k\pi)$

Here, $x(t)$ is (strictly) monotonic increasing in $x_0$ and (strictly) monotonic decreasing in $\dot{x}_0$ for positive values. Hence, the $\alpha$-cuts $x(\alpha, t) = [x_L(\alpha, t), x_R(\alpha, t)]$ of $\dot{x}(t)$ are

\[
x^L(\alpha, t) = x(x_0^L(\alpha), \dot{x}_0^L(\alpha), t) = (\bar{x}_0 - \tau_{x_0}^L(1 - \alpha)) \cos(\omega t) \\
+ (\bar{x}_0 - \tau_{x_0}^L(1 - \alpha)) \frac{\sin(\omega t)}{\omega}, \quad 0 < \alpha \leq 1,
\]

\[
x^R(\alpha, t) = x(x_0^R(\alpha), \dot{x}_0^R(\alpha), t) = (\bar{x}_0 + \tau_{x_0}^R(1 - \alpha)) \cos(\omega t) \\
+ (\bar{x}_0 + \tau_{x_0}^R(1 - \alpha)) \frac{\sin(\omega t)}{\omega}, \quad 0 < \alpha \leq 1.
\]

With

\[
x^L(0, t) = (\bar{x}_0 - \tau_{x_0}^L) \cos(\omega t) + (\bar{x}_0 + \tau_{x_0}^R) \frac{\sin(\omega t)}{\omega}, \quad x^L(0, t) < x(t) \leq x^L(1, t),
\]

\[
x^L(1, t) = \bar{x}_0 \cos(\omega t) + \bar{x}_0 \frac{\sin(\omega t)}{\omega} = x^R(1, t),
\]

\[
x^R(0, t) = (\bar{x}_0 + \tau_{x_0}^R) \cos(\omega t) + (\bar{x}_0 - \tau_{x_0}^L) \frac{\sin(\omega t)}{\omega}, \quad x^R(1, t) < x(t) < x^R(0, t).
\]
4.1.5 Illustrative example

As an illustrative example, we consider a system with the parameters $\bar{x}_0 = 1 \text{ cm}$, $\tau^L_{x_0} = \tau^R_{x_0} = 0.5 \text{ cm}$, $\ddot{x}_0 = 1.5 \text{ cm/s}$, $\tau^L_{\dot{x}_0} = 0.5 \text{ cm/s}$, $\tau^R_{\dot{x}_0} = 1 \text{ cm/s}$, $k = 0.05 \text{ N/m}$, and $m = 0.05 \text{ kg}$. The membership function of the fuzzy position $\tilde{x}(t)$ for $\omega t \in [0, 2\pi]$ is illustrated in Figure 3.

4.2 Weak damping

In case of $\xi^2 < \omega^2$, the general solution of Eq. (3) is given by [4]

$$x(t) = e^{-\xi t} \left( x_0 \cos(\omega_d t) + (\dot{x}_0 + \xi x_0) \frac{\sin(\omega_d t)}{\omega_d} \right),$$

where

$$\omega_d = \omega \sqrt{1 - \zeta^2}$$

denotes the damped natural frequency and

$$\zeta = \frac{\xi}{\omega} = \frac{d}{2\sqrt{mk}}$$

the damping ratio of the system.

Since for $k \in \mathbb{N}_0$

$$\frac{\partial x(t)}{\partial x_0} = \frac{e^{-\xi t} \omega \cos(\omega_d t - \varphi)}{\omega_d} \begin{cases} \geq 0, & \omega t \in [0 + 2k\pi, 0.5\pi + \varphi + 2k\pi] \cup [1.5\pi + \varphi + 2k\pi, 2\pi + 2k\pi], \\ < 0, & \omega t \in (0.5\pi + \varphi + 2k\pi, 1.5\pi + \varphi + 2k\pi), \end{cases}$$

$$\frac{\partial x(t)}{\partial \dot{x}_0} = \frac{e^{-\xi t} \sin(\omega_d t)}{\omega_d} \begin{cases} \geq 0, & \omega t \in [0 + 2k\pi, \pi + 2k\pi], \\ < 0, & \omega t \in (\pi + 2k\pi, 2\pi + 2k\pi) \end{cases}$$

with

$$\varphi = \arctan \left( \frac{\xi}{\omega_d} \right) = \arctan \left( \frac{\zeta}{\sqrt{1 - \zeta^2}} \right),$$

we can again distinguish between four domains.
4.2.1 \( \omega_d t \in [0 + 2k\pi, 0.5\pi + \varphi + 2k\pi] \)

Here, \( x(t) \) is (strictly) monotonic increasing in both \( x_0 \) and \( \dot{x}_0 \) for positive values. Hence, the \( \alpha \)-cuts \( x(\alpha, t) = [x^L(\alpha, t), x^R(\alpha, t)] \) of \( \dot{x}(t) \) are

\[
x^L(\alpha, t) = x(x^L_0(\alpha), x^L_0(\alpha), t) = e^{-\xi t} \left( (\bar{x}_0 - \tau^L_{x_0}(1 - \alpha)) \cos(\omega_d t) \right.
\]
\[
+ ((\bar{x}_0 - \tau^L_{x_0}(1 - \alpha)) + \xi(\bar{x}_0 - \tau^L_{x_0}(1 - \alpha))) \frac{\sin(\omega_d t)}{\omega_d} \right), \quad 0 < \alpha \leq 1,
\]
\[
x^R(\alpha, t) = x(x^R_0(\alpha), x^R_0(\alpha), t) = e^{-\xi t} \left( (\bar{x}_0 + \tau^R_{x_0}(1 - \alpha)) \cos(\omega_d t) \right.
\]
\[
+ ((\bar{x}_0 + \tau^R_{x_0}(1 - \alpha)) + \xi(\bar{x}_0 + \tau^R_{x_0}(1 - \alpha))) \frac{\sin(\omega_d t)}{\omega_d} \right), \quad 0 < \alpha \leq 1.
\]

With

\[
x^L(0, t) = e^{-\xi t} \left( (\bar{x}_0 - \tau^L_{x_0}) \cos(\omega_d t) + ((\bar{x}_0 - \tau^L_{x_0}) + \xi(\bar{x}_0 - \tau^L_{x_0})) \frac{\sin(\omega_d t)}{\omega_d} \right), \quad L,
\]
\[
x^L(1, t) = e^{-\xi t} \left( \bar{x}_0 \cos(\omega_d t) + (\bar{x}_0 + \xi \bar{x}_0) \frac{\sin(\omega_d t)}{\omega_d} = x^R(1, t), \right.
\]
\[
x^R(0, t) = e^{-\xi t} \left( (\bar{x}_0 + \tau^R_{x_0}) \cos(\omega_d t) + ((\bar{x}_0 + \tau^R_{x_0}) + \xi(\bar{x}_0 + \tau^R_{x_0})) \frac{\sin(\omega_d t)}{\omega_d} \right), \quad R,
\]

the membership function of \( \dot{x}(t) \) yields

\[
\mu_{\dot{x}(t)}(x(t)) = \begin{cases} \frac{-\omega_d(\bar{x}_0 - \tau^L_{x_0}) \cos(\omega_d t) - ((\bar{x}_0 - \tau^L_{x_0}) + \xi(\bar{x}_0 - \tau^L_{x_0})) \sin(\omega_d t) + e^{\xi t} \omega_d x(t)}{\omega_d \tau^L_{x_0} \cos(\omega_d t) + (\tau^L_{x_0} + \xi \tau^L_{x_0}) \sin(\omega_d t)}, & L, \\
\frac{\omega_d \tau^R_{x_0} \cos(\omega_d t) + (\tau^R_{x_0} + \xi \tau^R_{x_0}) \sin(\omega_d t)}{\omega_d \tau^R_{x_0} \cos(\omega_d t) + (\tau^R_{x_0} + \xi \tau^R_{x_0}) \sin(\omega_d t)}, & R, \end{cases}
\]

where

\[
L \equiv x^L(0, t) < x(t) \leq x^L(1, t),
\]
\[
R \equiv x^R(1, t) < x(t) < x^R(0, t).
\]

4.2.2 \( \omega_d t \in (0.5\pi + \varphi + 2k\pi, \pi + 2k\pi] \)

Here, \( x(t) \) is (strictly) monotonic decreasing in \( x_0 \) and (strictly) monotonic increasing in \( \dot{x}_0 \) for positive values. Hence, the \( \alpha \)-cuts \( x(\alpha, t) = [x^L(\alpha, t), x^R(\alpha, t)] \) of \( \dot{x}(t) \) are

\[
x^L(\alpha, t) = x(x^L_0(\alpha), x^L_0(\alpha), t) = e^{-\xi t} \left( (\bar{x}_0 + \tau^R_{x_0}(1 - \alpha)) \cos(\omega_d t) \right.
\]
\[
+ ((\bar{x}_0 + \tau^R_{x_0}(1 - \alpha)) + \xi(\bar{x}_0 + \tau^R_{x_0}(1 - \alpha))) \frac{\sin(\omega_d t)}{\omega_d} \right), \quad 0 < \alpha \leq 1,
\]
\[
x^R(\alpha, t) = x(x^R_0(\alpha), x^R_0(\alpha), t) = e^{-\xi t} \left( (\bar{x}_0 - \tau^L_{x_0}(1 - \alpha)) \cos(\omega_d t) \right.
\]
\[
+ ((\bar{x}_0 - \tau^L_{x_0}(1 - \alpha)) + \xi(\bar{x}_0 - \tau^L_{x_0}(1 - \alpha))) \frac{\sin(\omega_d t)}{\omega_d} \right), \quad 0 < \alpha \leq 1.
\]

With

\[
x^L(0, t) = e^{-\xi t} \left( (\bar{x}_0 + \tau^R_{x_0}) \cos(\omega_d t) + ((\bar{x}_0 - \tau^L_{x_0}) + \xi(\bar{x}_0 + \tau^R_{x_0})) \frac{\sin(\omega_d t)}{\omega_d} \right),
\]

\[
x^R(0, t) = e^{-\xi t} \left( (\bar{x}_0 - \tau^L_{x_0}) \cos(\omega_d t) + ((\bar{x}_0 + \tau^R_{x_0}) + \xi(\bar{x}_0 - \tau^L_{x_0})) \frac{\sin(\omega_d t)}{\omega_d} \right).
\]
$x^L(1, t) = e^{-\xi t} \left( \bar{x}_0 \cos(\omega_d t) + (\bar{x}_0 + \xi \bar{x}_0) \frac{\sin(\omega_d t)}{\omega_d} \right) = x^R(1, t)$,

$x^R(0, t) = e^{-\xi t} \left( (\bar{x}_0 - t^L_{x_0}) \cos(\omega_d t) + ((\bar{x}_0 + \tau^R_{x_0}) + \xi (\bar{x}_0 - t^L_{x_0})) \frac{\sin(\omega_d t)}{\omega_d} \right)$,

the membership function of $\hat{x}(t)$ yields

$$
\mu_{\hat{x}(t)}(x(t)) = \begin{cases} 
+\omega_d(\bar{x}_0 + \tau^R_{x_0}) \cos(\omega_d t) + ((\bar{x}_0 + \tau^R_{x_0}) + \xi (\bar{x}_0 + \tau^R_{x_0})) \sin(\omega_d t) - e^{\xi t}\omega_d x(t), & L, \\
-\omega_d(\bar{x}_0 - \tau^L_{x_0}) \cos(\omega_d t) - ((\bar{x}_0 - \tau^L_{x_0}) + \xi (\bar{x}_0 - \tau^L_{x_0})) \sin(\omega_d t) + e^{\xi t}\omega_d x(t), & R,
\end{cases}
$$

where again

$L \doteq x^L(0, t) < x(t) \leq x^L(1, t)$,

$R \doteq x^R(1, t) < x(t) < x^R(0, t)$.

### 4.2.3 $\omega_d t \in (\pi + 2k\pi, 1.5\pi + \varphi + 2k\pi]$]

Here, $x(t)$ is (strictly) monotonic decreasing in both $x_0$ and $\hat{x}_0$ for positive values. Hence, the $\alpha$-cuts $x(\alpha, t) = [x^L(\alpha, t), x^R(\alpha, t)]$ of $\hat{x}(t)$ are

$x^L(\alpha, t) = x(\alpha^L_0, \alpha^L_0, t) = e^{-\xi t} \left( (\bar{x}_0 + \tau^R_{x_0}(1 - \alpha)) \cos(\omega_d t) + ((\bar{x}_0 + \tau^R_{x_0}(1 - \alpha)) + \xi (\bar{x}_0 + \tau^R_{x_0}(1 - \alpha))) \frac{\sin(\omega_d t)}{\omega_d} \right), \quad 0 < \alpha \leq 1,$

$x^R(\alpha, t) = x(\alpha^R_0, \alpha^R_0, t) = e^{-\xi t} \left( (\bar{x}_0 - \tau^L_{x_0}(1 - \alpha)) \cos(\omega_d t) + ((\bar{x}_0 - \tau^L_{x_0}(1 - \alpha)) + \xi (\bar{x}_0 - \tau^L_{x_0}(1 - \alpha))) \frac{\sin(\omega_d t)}{\omega_d} \right), \quad 0 < \alpha \leq 1.$

With

$x^L(0, t) = e^{-\xi t} \left( (\bar{x}_0 + \tau^R_{x_0}) \cos(\omega_d t) + ((\bar{x}_0 + \tau^R_{x_0}) + \xi (\bar{x}_0 + \tau^R_{x_0})) \frac{\sin(\omega_d t)}{\omega_d} \right),$

$x^L(1, t) = e^{-\xi t} \left( \bar{x}_0 \cos(\omega_d t) + (\bar{x}_0 + \xi \bar{x}_0) \frac{\sin(\omega_d t)}{\omega_d} \right) = x^R(1, t),$

$x^R(0, t) = e^{-\xi t} \left( (\bar{x}_0 - \tau^L_{x_0}) \cos(\omega_d t) + ((\bar{x}_0 - \tau^L_{x_0}) + \xi (\bar{x}_0 - \tau^L_{x_0})) \frac{\sin(\omega_d t)}{\omega_d} \right),$

the membership function of $\hat{x}(t)$ yields

$$
\mu_{\hat{x}(t)}(x(t)) = \begin{cases} 
+\omega_d(\bar{x}_0 + \tau^R_{x_0}) \cos(\omega_d t) + ((\bar{x}_0 + \tau^R_{x_0}) + \xi (\bar{x}_0 + \tau^R_{x_0})) \sin(\omega_d t) - e^{\xi t}\omega_d x(t), & L, \\
-\omega_d(\bar{x}_0 - \tau^L_{x_0}) \cos(\omega_d t) - ((\bar{x}_0 - \tau^L_{x_0}) + \xi (\bar{x}_0 - \tau^L_{x_0})) \sin(\omega_d t) + e^{\xi t}\omega_d x(t), & R,
\end{cases}
$$

where again

$L \doteq x^L(0, t) < x(t) \leq x^L(1, t)$,

$R \doteq x^R(1, t) < x(t) < x^R(0, t)$.
4.2.4 \[ \omega_d t \in (1.5\pi + \varphi + 2k\pi, 2\pi + 2k\pi) \]

Here, \( x(t) \) is (strictly) monotonic increasing in \( x_0 \) and (strictly) monotonic decreasing in \( \dot{x} \) for positive values. Hence, the \( \alpha \)-cuts \( x(\alpha, t) = [x^L(\alpha, t), x^R(\alpha, t)] \) of \( \dot{x}(t) \) are

\[
x^L(\alpha, t) = x(x^L_0(\alpha), \dot{x}^R_0(\alpha), t) = e^{-\xi t}\left((\bar{x}_0 - \tau^L_{x_0}(1 - \alpha)) \cos(\omega_d t) + (\dot{x}_0 + \tau^R_{x_0}(1 - \alpha)) \right) + \xi(\bar{x}_0 - \tau^L_{x_0}(1 - \alpha)) \frac{\sin(\omega_d t)}{\omega_d}, \quad 0 < \alpha \leq 1,
\]

\[
x^R(\alpha, t) = x(x^R_0(\alpha), \dot{x}^L_0(\alpha), t) = e^{-\xi t}\left((\bar{x}_0 + \tau^L_{x_0}(1 - \alpha)) \cos(\omega_d t) + (\dot{x}_0 - \tau^R_{x_0}(1 - \alpha)) \right) + \xi(\bar{x}_0 + \tau^L_{x_0}(1 - \alpha)) \frac{\sin(\omega_d t)}{\omega_d}, \quad 0 < \alpha \leq 1.
\]

With

\[
x^L(0, t) = e^{-\xi t}\left((\bar{x}_0 - \tau^L_{x_0}) \cos(\omega_d t) + (\dot{x}_0 + \tau^R_{x_0}) + \xi(\bar{x}_0 - \tau^L_{x_0}) \frac{\sin(\omega_d t)}{\omega_d} \right),
\]

\[
x^L(1, t) = e^{-\xi t}\left((\bar{x}_0 \cos(\omega_d t) + (\dot{x}_0 + \xi\bar{x}_0) \frac{\sin(\omega_d t)}{\omega_d} \right) = x^R(1, t),
\]

\[
x^R(0, t) = e^{-\xi t}\left((\bar{x}_0 + \tau^R_{x_0}) \cos(\omega_d t) + (\dot{x}_0 - \tau^L_{x_0}) + \xi(\bar{x}_0 + \tau^R_{x_0}) \frac{\sin(\omega_d t)}{\omega_d} \right),
\]

the membership function of \( \dot{x}(t) \) yields

\[
\mu_{\dot{x}(t)}(x(t)) = \begin{cases} 
-\omega_d(\bar{x}_0 - \tau^L_{x_0}) \cos(\omega_d t) - ((\dot{x}_0 + \tau^R_{x_0}) + \xi(\bar{x}_0 - \tau^L_{x_0})) \frac{\sin(\omega_d t)}{\omega_d} + e^{\xi t}\omega_d x(t), & L, \\
\omega_d\tau^L_{x_0} \cos(\omega_d t) - (\tau^R_{x_0} - \xi\tau^L_{x_0}) \frac{\sin(\omega_d t)}{\omega_d}, & R,
\end{cases}
\]

where again

\[
L \doteq x^L(0, t) < x(t) \leq x^L(1, t),
\]

\[
R \doteq x^R(1, t) < x(t) < x^R(0, t).
\]

4.2.5 Illustrative example

As an illustrative example, we consider a system with the parameters from Section 4.1.5 and \( d = 0.05 \text{ kg/s} \). The membership function of the fuzzy position \( \dot{x}(t) \) for \( \omega_d \in [0, 2\pi] \) is illustrated in Figure 4.

4.3 Critical damping

In case of \( \xi^2 = \omega^2 \), the general solution of Eq. (3) is given by [4]

\[
x(t) = e^{-\xi t}(x_0 + (\dot{x}_0 + \xi x_0)t).
\]

Since

\[
\begin{align*}
\frac{\partial x(t)}{\partial x_0} &= e^{-\xi t}(1 + \xi t) > 0, \\
\frac{\partial x(t)}{\partial \dot{x}_0} &= e^{-\xi t} t \geq 0,
\end{align*}
\]
as an illustrative example, we consider a system with the parameters from section 4.1.5 and \( \tilde{x}(t) \). The membership function of the fuzzy position \( \tilde{x}(t) \) with weak damping

\( x(t) \) is (strictly) monotonic increasing in \( x_0 \) and monotonic increasing in \( \dot{x}_0 \) for positive values. Hence, the \( \alpha \)-cuts \( x(\alpha, t) = [x^L(\alpha, t), x^R(\alpha, t)] \) of \( \tilde{x}(t) \) are

\[
x^L(\alpha, t) = x(x^L_0(\alpha), x^R_0(\alpha), t) = e^{-\xi(t)}((\ddot{x}_0 - \tau^L_{x_0}(1 - \alpha))
+ ((\ddot{x}_0 - \tau^L_{x_0}(1 - \alpha)) + \xi(\ddot{x}_0 - \tau^L_{x_0}(1 - \alpha)))t), \quad 0 < \alpha \leq 1,
\]

\[
x^R(\alpha, t) = x(x^R_0(\alpha), x^R_0(\alpha), t) = e^{-\xi(t)}((\ddot{x}_0 + \tau^R_{x_0}(1 - \alpha))
+ ((\ddot{x}_0 + \tau^R_{x_0}(1 - \alpha)) + \xi(\ddot{x}_0 + \tau^R_{x_0}(1 - \alpha)))t), \quad 0 < \alpha \leq 1.
\]

With

\[
x^L(0, t) = e^{-\xi(t)}((\ddot{x}_0 - \tau^L_{x_0}) + ((\ddot{x}_0 - \tau^L_{x_0}) + \xi(\ddot{x}_0 - \tau^L_{x_0}))t),
\]

\[
x^L(1, t) = e^{-\xi(t)}((\ddot{x}_0 + \tau^L_{x_0}) + ((\ddot{x}_0 + \tau^L_{x_0}) + \xi(\ddot{x}_0 + \tau^L_{x_0}))t),
\]

\[
x^R(0, t) = e^{-\xi(t)}((\ddot{x}_0 + \tau^R_{x_0}) + ((\ddot{x}_0 + \tau^R_{x_0}) + \xi(\ddot{x}_0 + \tau^R_{x_0}))t),
\]

the membership function of \( \tilde{x}(t) \) yields

\[
\mu_{\tilde{x}(t)}(x(t)) = \begin{cases} 
\frac{-(\ddot{x}_0 - \tau^L_{x_0}) - ((\ddot{x}_0 - \tau^L_{x_0}) + \xi(\ddot{x}_0 - \tau^L_{x_0}))t + e^{\xi(t)}x(t)}{\tau^L_{x_0} + (\tau^L_{x_0} + \xi\tau^L_{x_0})t}, & x^L(0, t) < x(t) \leq x^L(1, t), \\
\frac{+(\ddot{x}_0 + \tau^R_{x_0}) + ((\ddot{x}_0 + \tau^R_{x_0}) + \xi(\ddot{x}_0 + \tau^R_{x_0}))t - e^{\xi(t)}x(t)}{\tau^R_{x_0} + (\tau^R_{x_0} + \xi\tau^R_{x_0})t}, & x^R(1, t) < x(t) < x^R(0, t).
\end{cases}
\]

As an illustrative example, we consider a system with the parameters from section 4.1.5 and \( d = 0.1 \text{ kg/s} \). The membership function of the fuzzy position \( \tilde{x}(t) \) for \( t \in [0, 8] \text{ s} \) is illustrated in Figure 5.

### 4.4 Strong damping

In case of \( \xi^2 > \omega^2 \), the general solution of Eq. (3) is given by \([4]\)

\[
x(t) = e^{-\xi(t)}(x_0 \cosh(\mu t) + (\dot{x}_0 + \xi x_0)\frac{\sinh(\mu t)}{\mu}),
\]

where

\[
\mu = \omega \sqrt{\xi^2 - 1}
\]

can be viewed as the strong damped natural frequency.
Since
\[
\begin{align*}
\frac{\partial x(t)}{\partial x_0} & = e^{-\xi t} \omega \sinh(\mu t + \psi) > 0, \\
\frac{\partial x(t)}{\partial \dot{x}_0} & = \frac{e^{-\xi t} \sinh(\mu t)}{\mu} \geq 0,
\end{align*}
\]
with
\[
\psi = \text{artanh}\left(\frac{\mu}{\xi}\right) = \text{artanh}\left(\frac{\sqrt{\xi^2 - 1}}{\xi}\right),
\]
\(x(t)\) is (strictly) monotonic increasing in \(x_0\) and monotonic increasing in \(\dot{x}_0\) for positive values. Hence, the \(\alpha\)-cuts \(x(\alpha, t) = [x^L(\alpha, t), x^R(\alpha, t)]\) of \(\dot{x}(t)\) are
\[
x^L(\alpha, t) = x(x^L_0(\alpha), \dot{x}^L_0(\alpha), t) = e^{-\xi t} \left( (\bar{x}_0 - \tau^L_{x_0}(1 - \alpha)) \cosh(\mu t) \\
+ \left( (\bar{x}_0 - \tau^L_{\dot{x}_0}(1 - \alpha)) + \xi (\bar{x}_0 - \tau^L_{x_0}(1 - \alpha)) \frac{\sinh(\mu t)}{\mu} \right) \right), \quad 0 < \alpha \leq 1,
\]
\[
x^R(\alpha, t) = x(x^R_0(\alpha), \dot{x}^R_0(\alpha), t) = e^{-\xi t} \left( (\bar{x}_0 + \tau^R_{x_0}(1 - \alpha)) \cosh(\mu t) \\
+ \left( (\bar{x}_0 + \tau^R_{\dot{x}_0}(1 - \alpha)) + \xi (\bar{x}_0 + \tau^R_{x_0}(1 - \alpha)) \frac{\sinh(\mu t)}{\mu} \right) \right), \quad 0 < \alpha \leq 1.
\]
With
\[
x^L(0, t) = e^{-\xi t} \left( (\bar{x}_0 - \tau^L_{x_0}) \cosh(\mu t) + \left( (\bar{x}_0 - \tau^L_{x_0}) + \xi (\bar{x}_0 - \tau^L_{x_0}) \right) \frac{\sinh(\mu t)}{\mu} \right),
\]
\[
x^L(1, t) = e^{-\xi t} \left( \bar{x}_0 \cosh(\mu t) + \left( \bar{x}_0 + \xi \bar{x}_0 \right) \frac{\sinh(\mu t)}{\mu} \right),
\]
\[
x^R(0, t) = e^{-\xi t} \left( (\bar{x}_0 + \tau^R_{x_0}) \cosh(\mu t) + \left( (\bar{x}_0 + \tau^R_{x_0}) + \xi (\bar{x}_0 + \tau^R_{x_0}) \right) \frac{\sinh(\mu t)}{\mu} \right),
\]
the membership function of \(\dot{x}(t)\) yields
\[
\mu_{\dot{x}(t)}(x(t)) = \begin{cases} 
-\mu(\bar{x}_0 - \tau^L_{x_0}) \cosh(\mu t) - (\bar{x}_0 - \tau^L_{x_0}) + \xi (\bar{x}_0 - \tau^L_{x_0}) \sinh(\mu t) + e^{\xi t} \mu x(t), & L, \\
\mu \tau^L_{x_0} \cosh(\mu t) + (\tau^L_{x_0} - \xi \tau^L_{x_0}) \sinh(\mu t), & \\
+\mu(\bar{x}_0 + \tau^R_{x_0}) \cosh(\mu t) + (\bar{x}_0 + \tau^R_{x_0}) + \xi (\bar{x}_0 + \tau^R_{x_0}) \sinh(\mu t) - e^{\xi t} \mu x(t), & R, \\
\mu \tau^R_{x_0} \cosh(\mu t) + (\tau^R_{x_0} - \xi \tau^R_{x_0}) \sinh(\mu t), &
\end{cases}
\]
Figure 6: Membership function of the fuzzy position $\tilde{x}(t)$ with strong damping

where

\[
L \triangleq x^L(0, t) < x(t) \leq x^L(1, t),
\]
\[
R \triangleq x^R(1, t) < x(t) < x^R(0, t).
\]

As an illustrative example, we consider a system with the parameters from Section 4.1.5 and $d = 0.15 \text{ kg/s}$. The membership function of the fuzzy position $\tilde{x}(t)$ for $\mu t \in [0, 2\pi]$ is illustrated in Figure 6.

5 Conclusions

We introduced closed-form symbolic expressions for the membership function of the fuzzy position of a one degree of freedom, linear free vibrating mass system with fuzzy initial conditions. This relieves the engineer from the burden of propagating the uncertainties through the computations to obtain the uncertain output. The solutions can be used, e.g., in more complex simulations to save computation time.

References


