The Fourier Transform

• Fourier Series

• Fourier Transform

• The Basic Theorems and Applications

• Sampling


Fourier Series

- Fourier series is an expansion (German: Entwicklung) of a periodic function \( f(x) \) (period \( 2L \), angular frequency \( \pi/L \)) in terms of a sum of sine and cosine functions with angular frequencies that are integer multiples of \( \pi/L \).

- Any piecewise continuous function \( f(x) \) in the interval \([-L,L]\) may be approximated with the mean square error converging to zero.
Fourier Series

- Sine and cosine functions \( \{\sin(nx), \cos(nx)\} \) form a complete orthogonal system over \([-\pi, \pi]\)
- Basic relations \((m, n \neq 0)\):
  \[
  \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \pi \delta_{mn}
  \]
  \[
  \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \pi \delta_{mn}
  \]
  \[
  \int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0
  \]
  \[
  \int_{-\pi}^{\pi} \sin(mx)\,dx = 0
  \]
  \[
  \int_{-\pi}^{\pi} \cos(mx)\,dx = 0
  \]
- \(\delta_{mn}\): Kronecker delta
  - \(\delta_{mn} = 0\) for \(m \neq n\), \(\delta_{mn} = 1\) for \(m = n\)
Fourier Series: Sawtooth

- Example: Sawtooth wave
Fourier Series

- Fourier series is given by
  \[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \]

- where
  \[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx \]
  \[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx \]
  \[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx \]

- If the function \( f(x) \) has a finite number of discontinuities and a finite number of extrema (Dirichlet conditions): The Fourier series converges to the original function at points of continuity or to the average of the two limits at points of discontinuity.
Fourier Series

• For a function $f(x)$ periodic on an interval $[-L,L]$ instead of $[-\pi,\pi]$ change of variables can be used to transform the interval of integration

$$x \equiv \frac{\pi x'}{L} \quad \text{and} \quad dx \equiv \frac{\pi dx'}{L}$$

• Then

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n \pi x'}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n \pi x'}{L}\right)$$

• and

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x') dx' \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x') \cos\left(\frac{n \pi x'}{L}\right) dx' \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x') \sin\left(\frac{n \pi x'}{L}\right) dx'$$
Fourier Series

• For a function $f(x)$ periodic on an interval $[0,2L]$ instead of $[-\pi,\pi]$:

\[
a_n = \frac{1}{L} \int_{0}^{2L} f(x') dx'\]

\[
a_n = \frac{1}{L} \int_{0}^{2L} f(x') \cos \left( \frac{n\pi x'}{L} \right) dx'
\]

\[
b_n = \frac{1}{L} \int_{0}^{2L} f(x') \sin \left( \frac{n\pi x'}{L} \right) dx'
\]
Fourier Series: Sawtooth

- Example: Sawtooth wave

\[ f(x) = \frac{x}{2L} \]

\[ a_0 = \frac{1}{L} \int_0^{2L} x \, dx = 1 \]

\[ a_n = \frac{1}{L} \int_0^{2L} x \cos \left( \frac{n\pi x}{L} \right) \, dx \]

\[ = \frac{2n\pi \cos(n\pi) - \sin(n\pi)}{n^2 \pi^2} \sin(n\pi) \]

\[ = 0 \]

\[ b_n = \frac{1}{L} \int_0^{2L} x \sin \left( \frac{n\pi x}{L} \right) \, dx \]

\[ = -\frac{2n\pi \cos(2n\pi) + \sin(2n\pi)}{2n^2 \pi^2} \]

\[ = -\frac{1}{n\pi} \]
Fourier Series: Sawtooth

- Example: Sawtooth wave
  \[ f(x) = \frac{x}{2L} \]
- Fourier series:
  \[ g(m, x) = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{m} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) \]
Fourier Series: Sawtooth

\[ x / (2L) \]

\[ \omega / (\pi / L) \]

Amplitude $a_0/2, b_n$
Example: Approximation by Fourier-Series

\[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{m} a_n \cos(nx) + \sum_{n=1}^{m} b_n \sin(nx) \]
Approximation by Fourier-Series

\[ m = 7 \]

\[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{m} a_n \cos(nx) + \sum_{n=1}^{m} b_n \sin(nx) \]
Approximation by Fourier-Series

\( m=10 \)

\[
f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{m} a_n \cos(nx) + \sum_{n=1}^{m} b_n \sin(nx)
\]
Fourier Series: Properties

- Around points of discontinuity, a "ringing“, called Gibbs phenomenon occurs

- If a function $f(x)$ is even, $f(x)\sin(nx)$ is odd and thus $\int_{-\pi}^{\pi} f(x)\sin(nx)dx = 0$

  ⇒ Even function: $b_n = 0$ for all $n$

- If a function $f(x)$ is odd, $f(x)\cos(nx)$ is odd and thus $\int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0$

  ⇒ Odd function: $a_n = 0$ for all $n$
Complex Fourier Series

• Fourier series with complex coefficients

\[ f(x) = \sum_{n=-\infty}^{\infty} A_n e^{inx} \]

• Calculation of coefficients:

\[ A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx \]

• Coefficients may be expressed in terms of those in the real Fourier series

\[ A_n = \frac{1}{2} \left( a_n + ib_n \right) \text{ for } n < 0 \]

\[ A_n = \frac{1}{2} a_0 \text{ for } n = 0 \]

\[ A_n = \frac{1}{2} \left( a_n - ib_n \right) \text{ for } n > 0 \]
Fourier Transform

• Generalization of the complex Fourier series for infinite $L$ and continuous variable $k$
  - $L \to \infty$, $n/L \to k$, $A_n \to F(k)dk$

• Forward transform
  $$F(k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx} \, dx \quad \text{or} \quad F(k) = F_x(f(x))(k)$$

• Inverse transform
  $$f(x) = \int_{-\infty}^{\infty} F(k)e^{2\pi ikx} \, dk \quad \quad f(x) = F_k^{-1}(F(k))(x)$$

• Symbolic notation
  $$f(x) \longrightarrow F(k)$$
Forward and Inverse Transform

• If $\int_{-\infty}^{\infty} |f(x)| \, dx$ exists
  1. $f(x)$ has a finite number of discontinuities
  2. $f(x)$ has a finite number of discontinuities
  3. The variation of the function is bounded

• Forward and inverse transform:

• For $f(x)$ continuous at $x$ $f(x) = F_k^{-1}(F_x f(x))(x) = \int_{-\infty}^{\infty} e^{2\pi ikx} \left( \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx} \, dx \right) \, dk$

• For $f(x)$ discontinuous at $x$ $F_k^{-1}(F_x f(x))(x) = \frac{1}{2} (f(x_+) + f(x_-))$
Fourier Cosine Transform and Fourier Sine Transform

• Any function may be split into an even and an odd function

\[ f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] = E(x) + O(x) \]

• Fourier transform may be expressed in terms of the Fourier cosine transform and Fourier sine transform

\[ F(k) = \int_{-\infty}^{\infty} E(x) \cos(2\pi k x) \, dx - i \int_{-\infty}^{\infty} O(x) \sin(2\pi k x) \, dx \]
Real Even Function

• If a real function \( f(x) \) is even, \( O(x) = 0 \)

\[
F(k) = \int_{-\infty}^{\infty} E(x) \cos(2\pi k x) dx = \int_{-\infty}^{\infty} f(x) \cos(2\pi k x) dx = 2 \int_{0}^{\infty} f(x) \cos(2\pi k x) dx
\]

• If a function is real and even, the Fourier transform is also real and even

Real Odd Function

- If a real function $f(x)$ is odd, $E(x) = 0$

$$F(k) = -i \int_{-\infty}^{\infty} O(x) \sin(2\pi k x) dx = -i \int_{-\infty}^{\infty} f(x) \sin(2\pi k x) dx = -2i \int_{0}^{\infty} f(x) \sin(2\pi k x) dx$$

- If a function is real and odd, the Fourier transform is imaginary and odd
### Symmetry Properties

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$F(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real and even</td>
<td>Real and even</td>
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<tr>
<td>Real and odd</td>
<td>Imaginary and odd</td>
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<td>Imaginary and odd</td>
<td>Real and odd</td>
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<tr>
<td>Real asymmetrical</td>
<td>Complex hermitian</td>
</tr>
<tr>
<td>Imaginary asymmetrical</td>
<td>Complex antihermitian</td>
</tr>
<tr>
<td>Even</td>
<td>Even</td>
</tr>
<tr>
<td>Odd</td>
<td>Odd</td>
</tr>
</tbody>
</table>

Hermitian: $f(x) = f(-x)$  
Antihermitian: $f(x) = -f^*(-x)$
Symmetry Properties

$f(x)$

Even
Odd
Real (Displaced to right)
Imaginary (Displaced to right)

$F(k)$

Even
Odd
Hermitian
Antihermiteian

Fourier Transform: Properties and Basic Theorems

- Linearity
  \[ af(x) + bg(x) \rightarrow aF(k) + bG(k) \]

- Shift theorem
  \[ f(x - x_0) \rightarrow e^{-2\pi i x_0 k} F(k) \]

- Modulation theorem
  \[ f(x) \cos(2\pi k_0 x) \rightarrow \frac{1}{2} (F(k - k_0) + F(k + k_0)) \]

- Derivative
  \[ f'(x) \rightarrow i2\pi k F(k) \]
Fourier Transform: Properties and Basic Theorems

- Parseval's / Rayleigh's theorem
  \[ \int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |F(k)|^2 \, dk \]

- Fourier Transform of the complex conjugated
  \[ f^*(x) \rightarrow F^*(-k) \]

- Similarity (a \neq 0 : real constant)
  \[ f(ax) \rightarrow |a|^{-1} F\left(\frac{k}{a}\right) \]
  - Special case (a = -1): \[ f(-x) \rightarrow F(-k) \]
Similarity Theorem

\[ f(x) \]

\[ f(2x) \]

\[ f(5x) \]

\[ F(k) \]

\[ \frac{1}{2} F\left(\frac{k}{2}\right) \]

\[ \frac{1}{5} F\left(\frac{k}{5}\right) \]
Convolution

- Convolution describes an action of an observing instrument (linear instrument) when it takes a weighted mean of a physical quantity over a narrow range of a variable
  - Examples:
    - Photo camera: “Measured” photo is described by real image convolved with a function describing the apparatus
    - Spectrometer: Measured spectrum is given by real spectrum convolved with a function describing limited resolution of the spectrometer

- Other terms for convolution: Folding (German: Faltung), composition product

- System theory: Linear time (or space) invariant systems are described by convolution
Convolutions

- Definition of convolution

\[ y(x) = f * g = \int_{-\infty}^{\infty} f(\tau) \cdot g(x - \tau) d\tau \]

- JAVA Applets: http://www.jhu.edu/~signals/


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Properties of the Convolution

• Properties of the convolution

\[ f \ast g = g \ast f \]  
\text{commutativity}

\[ f \ast (g \ast h) = (f \ast g) \ast h \]  
\text{associativity}

\[ f \ast (g + h) = (f \ast g) + (f \ast h) \]  
\text{distributivity}

\[ a(f \ast g) = (af) \ast g = f \ast (ag) \]

\[ \frac{d}{dx} \left( f \ast g \right) = \frac{df}{dx} \ast g = f \ast \frac{dg}{dx} \]

• The integral of a convolution is the product of integrals of the factors

\[ \int_{-\infty}^{\infty} (f \ast g)dx = \int_{-\infty}^{\infty} f(x)dx \int_{-\infty}^{\infty} g(x)dx \]
The Basic Theorems: Convolution

• The Fourier transform of the convolution of \( f(x) \) and \( g(x) \) is given by the product of the individual transforms \( F(k) \) and \( G(k) \)

\[
f(x) * g(x) \quad \longrightarrow \quad F(k)G(k)
\]

\[
f(x) * g(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i k x} f(x') g(x - x') dx' dx
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ e^{-2\pi i k x} f(x') dx' \right] \left[ e^{-2\pi i k(x - x')} g(x - x') dx \right]
= \left[ \int_{-\infty}^{\infty} e^{-2\pi i k x'} f(x') dx' \right] \left[ \int_{-\infty}^{\infty} e^{-2\pi i k x''} g(x'') dx'' \right]
= F(k)G(k)
\]

• Convolution in the spectral domain

\[
f(x)g(x) \quad \longrightarrow \quad F(k) * G(k)
\]
Cross-Correlation

- Definition of cross-correlation

\[ r_{fg}(t) = f(t) \star g(t) = \int_{-\infty}^{\infty} f^*(\tau) g(t + \tau) d\tau \]

\[ f(t) \star g(t) = f^*(-t) \star g(t) \]

- \( f \star g \neq g \star f \)

- Fourier transform:

\[ f(t) \star g(t) = f^*(-t) \star g(t), \quad f^*(-t) \longrightarrow F^*(k) \] (see FT rules)

\[ \Rightarrow f(t) \star g(t) \longrightarrow F^*(k)G(k) \]
Example: Cross-Correlation
Radar (Low Noise)

Signal sent by radar antenna

Signal received by radar antenna

Cross correlation of the signals
Example: Cross-Correlation

Radar

Signal sent by radar antenna

Signal received by radar antenna

Cross correlation of the signals
Example: Cross-Correlation

Ultrasonic Distance Measurement

Signal sent by ultrasonic transducer

Signal received by ultrasonic transducer

Cross correlation of the two signals

\[ d = \frac{vt_{\text{flight}}}{2} \quad (\Theta \approx 0) \]
Autocorrelation Theorem

- Autocorrelation

\[ f(t) \star f(t) = \int_{-\infty}^{\infty} f^*(\tau)f(t + \tau)d\tau \]

- Autocorrelation theorem:

\[ f(t) \star f(t) \rightarrow |F(k)|^2 \]

- Proof

\[ f(t) \star f(t) = f^*(-t) \star f(t), \quad f^*(-t) \rightarrow F^*(k) \]

\[ f(t) \star f(t) \rightarrow F^*(k)F(k) = |F(k)|^2 \]
Example: Autocorrelation
Detection of a Signal in the Presence of Noise

Signal: Noise

Autocorrelation function:
Peak at $t = 0$ : Noise
Example: Autocorrelation
Detection of a Signal in the Presence of Noise

Signal: Sinusoid plus noise

Autocorrelation function:
Peak at $t = 0$ : Noise
Period of $r_{ff}$ reveals that signal is periodic
Convolution

\[ f \star g \]

Cross-correlation

\[ f \star g \]

Autocorrelation

\[ f \star f \]

The Delta Function

- Function for the description of point sources, point masses, point charges, surface charges etc.
- Also called impulse function: Brief (unit area) impulse so that all measuring equipment is unable to resolve pulse
- Important attribute is not value at a given time (or position) but the integral

- Definition of the delta function (distribution, also called Dirac function)
  1. \( \delta(x) = 0 \) for \( x \neq 0 \)
  2. \( \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \)
The Delta Function

- The sifting (sampling) property of the delta function

\[ \int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0) \]

- Sampling of \( f(x) \) at \( x = a \)

\[ \int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a) \]

- Fourier transform:

\[ \delta(x-x_0) \cdot \int_{-\infty}^{\infty} \delta(x-x_0)e^{-2\pi ikx} \ dx = e^{-2\pi ikx_0} \]
Fourier Transform Pairs

- Cosine function

\[ \cos(\pi x) \]

\[ \Pi(x) = \frac{1}{2} \left[ \delta\left(x + \frac{1}{2}\right) + \delta\left(x - \frac{1}{2}\right) \right] \]

- Sine function

\[ \sin(\pi x) \]

\[ I_1(x) = \frac{1}{2} \left[ \delta\left(x + \frac{1}{2}\right) - \delta\left(x - \frac{1}{2}\right) \right] \]

\[ i I_1(x) \]
Fourier Transform Pairs

• Gaussian function

\[ e^{-\pi x^2} \]

• Rectangle function of unit height and base \( \Pi(x) \)

\[
\Pi(x) = \begin{cases} 
1 & |x| < \frac{1}{2} \\
\frac{1}{2} & |x| = \frac{1}{2} \\
0 & |x| > \frac{1}{2} 
\end{cases}
\]

\[
sinc(x) = \frac{\sin \pi x}{\pi x}
\]

\[
sinc(k)
\]
Fourier Transform Pairs

- Triangle function of unit height and area

\[ \Lambda(x) = \begin{cases} 
1 - |x| & |x| \leq 1 \\ 
0 & |x| > 1 
\end{cases} \]

- Constant function (unit height)

\[ \delta(k) \]
Sampling

- Sampling: “Noting” a function at finite intervals

Sampling of a Signal

Signal 1

Samples taken at equal intervals of time

Sampled signal
Sampling

Signal 1 + Signal 2

Samples taken at equal intervals of time

Sampled signal does not contain the additional Signal

Signal 2 (very short)
Aliasing

Sampling with a low sampling frequency

Sampled signal

Apparent signal
Sampling

- Sampling function (sampling comb) \( \text{III}(x) \) “Shah”

\[
\text{III}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)
\]

\[
\text{III}(ax) = \frac{1}{|a|} \sum_{n=-\infty}^{\infty} \delta\left(x - \frac{n}{a}\right)
\]

- Multiplication of \( f(x) \) with \( \text{III}(x) \) describes sampling at unit intervals

\[
\text{III}(x)f(x) = \sum_{n=-\infty}^{\infty} f(n)\delta(x - n)
\]

- Sampling at intervals \( \tau \):

\[
\text{III}\left(\frac{x}{\tau}\right)f(x) = \tau \sum_{n=-\infty}^{\infty} f(n\tau)\delta(x - n\tau)
\]

Sampling

- Fourier transform of the sampling function

\[
\begin{align*}
\mathcal{I}(x) & \quad \longrightarrow \quad \mathcal{I}(k) \\
\frac{1}{\tau} \mathcal{I}\left(\frac{x}{\tau}\right) & \quad \longrightarrow \quad \mathcal{I}\left(\tau k\right)
\end{align*}
\]

Sampling
Band limited signal

- Band limited signal: $F(k) = 0$ for $|k| > k_c$

Sampling in the Two Domains

\[ f(x) \]

\[ x \]

\[ \tau \]

\[ \text{III} \left( \frac{x}{\tau} \right) \]

\[ \Rightarrow \]

\[ \text{III} \left( \frac{x}{\tau} \right) f(x) \]

\[ F(k) \]

\[ k_c \]

\[ \ast \]

\[ \tau \text{III}(\tau k) \]

\[ k_c \]

\[ \Rightarrow \]

\[ \tau \text{III}(\tau k) \ast F(k) \]

\[ k_c \]

Reconstruction of the Signal

Convolution

Multiplication with rectangular function in order to remove additional signal components

Original signal in frequency domain

Fourier Transform
Critical Sampling and Undersampling

Sampling

- **Sampling theorem**
  - A function whose Fourier transform is zero \((F(k) = 0)\) for \(|k|>k_c\) is **fully specified** by values spaced at equal intervals \(\tau < 1/(2k_c)\).

- **Alternative (simplified) statement**: The sampling frequency must be higher than twice the highest frequency in the signal
  - Remark: This does not imply that a reconstruction is impossible for all cases if the sampling frequency is lower. If additional knowledge about the signal is available (e.g. lower limit of Fourier transform, single frequency signal), reconstruction may be possible.

- **Description of sampling at intervals \(\tau\)**: Multiplication of \(f(x)\) with \(\Pi(x/\tau)\)

- **Reconstruction of the signal**: Multiplication of the Fourier transform \((\tau \Pi(\varpi) \ast F(k))\) with \(\Pi\left(\frac{k}{2k_c}\right)\) and inverse transformation
  - Or: Convolution of the sampled function with a sinc-function