Robust Stability Analysis of Interconnected Systems with Uncertain Time-Varying Time Delays via IQCs

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Abstract—This paper presents a general modeling framework for interconnected LPV systems, that includes model classes like decomposable systems as special cases. The framework allows to consider arbitrary dynamic interconnection operators in the model. We propose to use integral quadratic constraints (IQCs) for robust stability analysis of such interconnected systems and provide convex sufficient conditions for the practically relevant case when communication between subsystems is delayed by an uncertain time-varying time delay. For that purpose, a survey of results on IQCs for time delay operators is presented. When analyzing robust stability, we distinguish between dynamic interconnection in- and output operators of the subsystems and a time-varying real valued interconnection topology, which may be switching arbitrarily fast under certain conditions.

I. INTRODUCTION

Interconnected systems are an active field of research that is receiving increasing attention. Its applications vary from networked systems, spatially-distributed systems [1] to multi-agent systems [2]. In [3] a very general approach for their modeling is proposed. A specialized modeling framework for distributed systems with identical subsystems is introduced in [4]. Motivated by [3], here we propose a modeling framework, which can yield more compact representations. It is shown, that this more compact form directly leads to the definition of decomposable LPV systems [4], [5] as a special case.

A feature of the interconnected systems considered here is that the interconnection between subsystems can encompass any static or dynamic uncertainty, e.g. parametric uncertainties, uncertain time delays or unmodelled saturations. The well-known framework of IQCs [6] can deal with this rich class of uncertainties. Based on [7], this paper presents analysis results for robust stability of interconnected systems via IQCs. It is focused on interconnected systems with time-delayed interconnections, where it is possible to distinguish between delays in the output as well as the input of each subsystem. This permits flexibility in modeling the interconnection operator with physical interpretation. The existing literature offers an extensive catalogue of IQCs for time delays, ranging from time-invariant delays [6], [8], [9], [10] to time-varying delays [11]. Here we focus on the case of uncertain time-varying delays, which is highly relevant, e.g. in the setting of multi-agent systems.

Outline: Section II reviews the framework of IQCs, with focus on time delays, and their usage for robust stability analysis. In Section III a general modeling framework for interconnected systems is introduced; special cases of practical interest are further explored. Section IV applies the theory of robust stability analysis via IQCs to the introduced class of interconnected systems with the focus on time-delayed interconnections. Numerical results are given in Section V. Finally the paper is concluded in Section VI.

II. PRELIMINARIES

A. IQCs

Integral Quadratic Constraints (IQCs) are defined as follows [7]:

Definition 1: Given a measurable Hermitian-valued function $\Pi : \mathbb{R} \to \mathbb{C}^{(l+m) \times (l+m)}$, the bounded causal operator $\Delta : \mathbb{L}_2^\infty [0, \infty) \to \mathbb{L}_1^\infty [0, \infty)$ is said to satisfy the IQC defined by $\Pi$ if

$$\int_{-\infty}^{\infty} \left[ \tilde{w}(j\omega) \hat{v}(j\omega)^* \right] \Pi(j\omega) \left[ \tilde{w}(j\omega) \hat{v}(j\omega) \right] d\omega \leq 0$$

(1)

with the signals $w \in \mathbb{L}_2^\infty [0, \infty)$ and $v = \Delta(w) \in \mathbb{L}_1^\infty [0, \infty)$.

With the IQC description the following stability condition for the feedback loop in Fig. 1 can be expressed [6].

Lemma 1: The feedback interconnection in Fig. 1 with $G(s) \in \mathbb{R} H^\infty_{l \times m}$ and bounded causal operator $\Delta$ is stable if

(i) for every $\tau \in [0, 1]$ the interconnection of $G(s)$ and $\tau \Delta$ is well-posed;

(ii) for every $\tau \in [0, 1]$ the IQC defined by $\Pi$ is satisfied by $\tau \Delta$ for $w$ and $v = \Delta(w)$;

(iii) there exists $\epsilon > 0$ such that

$$\left[ \begin{array}{c} G(j\omega) \\ I \end{array} \right] \Pi(j\omega) \left[ \begin{array}{c} G(j\omega) \\ I \end{array} \right]^* \leq -\epsilon I, \quad \forall \omega \in \mathbb{R}. \quad (2)$$

For rational $\Pi$, (2) can be transformed into a matrix inequality by the Kalman-Yakubovic-Lemma [12] and can thus be solved efficiently. Finding an appropriate parameterization of an IQC, that satisfies (ii) of Lemma 1, however, is not trivial. The task is to find a set of $\Pi$ that fulfills (1) for a given operator $\Delta$. Various IQC parameterizations for different types of operators have been proposed in the literature.
1) IQCs for Uncertain Time-Invariant Time Delays:
Assume $\Delta_r$ defines an unknown time-varying time delay
\[
\Delta_r(w) = w(t - \tau(t))
\]
with $0 \leq \tau(t) \leq \tau_0, |\dot{\tau}(t)| \leq d, \forall t \geq 0$. (3)

In frequency domain, a time delay can be represented in the complex plane as an arc of the unit circle, whose angle is defined by the maximal possible delay. IQCs for time delays are based on this graphical interpretation and are an approximation of the sector transformation as discussed in [6], [8]. In [9] it is shown how to turn multipliers of the IQC defined in [6], [8] into a general multiplier $\Pi(j\omega)$ with more degrees of freedom.

This frequency dependent non-rational multiplier $\Pi(j\omega)$, that switches unsmoothly with frequency is difficult to use for solving (2). There are different ways to deal with this problem. In [8] frequency gridding is proposed. But there is always a risk left that a crucial frequency is missed, such that robustness can hardly be predicted. In [6] a rational continuous approximation is used. In [10] a rational but sweeping approximation similar to the Padé approximation is employed. The sweeping problem is tackled by a generalized form of the Kalman-Yakubovich-Popov (KYP) Lemma for finite frequencies [13]. With this, the transformation of (2) with a frequency sweeping $\Pi(j\omega)$ into an LMI condition is possible.

Only unknown but constant time delays have been considered so far. In [11] uncertain time-varying time delays are considered for operators $\Delta_r$ as defined in (3). Here bounds on the operator are given for different values of $d$ and the required IQCs are listed. In the following, two IQC parameterizations are presented, which are particularly attractive due to their ease of implementation while leading to reasonable results. A more extensive list can be found in [11].

2) IQCs for Uncertain Time-Varying Time Delays:
For slowly varying delays the following result is given.

**Proposition 1 ([11]):** Suppose $|\dot{\tau}| \leq \delta < 1$, then $\Delta_r$ satisfies (1) for
\[
\Pi_{r_1} = \begin{bmatrix}
\frac{1}{1-d} X_{\tau_1} & 0 \\
0 & -X_{\tau_1}
\end{bmatrix}
\]
with $X_{\tau_1} = X_{\tau_1}^T \geq 0$.

**Proposition 2 ([11]):** Suppose $\tau(t) \in [0, \tau_0]$ and $|\dot{\tau}| \leq \delta < 2$ for all $t$, then $\Delta_r$ satisfies (1) for
\[
\Pi_{r_2} = \begin{bmatrix}
(|\psi(j\omega)|^2 - 1)X_{\tau_2} & X_{\tau_2} \\
X_{\tau_2}^T & -X_{\tau_2}
\end{bmatrix}
\]
with $X_{\tau_2} = X_{\tau_2}^T \geq 0$ and any bounded rational transfer function $\psi(j\omega)$ satisfying
\[
|\psi(j\omega)| \geq g(\omega) + \delta \quad \forall \omega \in \mathbb{R}
\]
where $\delta \in \mathbb{R}^+$ and
\[
g(\omega) = \begin{cases}
\sqrt{\frac{8}{2-d}} \left| \sin \left( \frac{\omega \tau_0}{2} \right) \right| & |\omega| \leq \frac{\pi}{\tau_0} \\
\sqrt{\frac{8}{2-d}} & |\omega| > \frac{\pi}{\tau_0}
\end{cases}
\]
In [11] the IQC is given for the operator $\Delta_r(w) = w(t) - w(t - \tau(t)) = w(t) - \Delta_r(w)$ as
\[
\hat{\Pi}_{r_2} = \begin{bmatrix}
|\psi(j\omega)|^2X_{\tau_2} & 0 \\
0 & -X_{\tau_2}
\end{bmatrix}
\]

It can be easily shown that this is equivalent to (4) by
\[
[s] \hat{\Pi}_{r_2} \begin{bmatrix} w \\
\Delta_r(w)
\end{bmatrix} = [s] \hat{\Pi}_{r_2} \begin{bmatrix} w \\
\Delta_r(w)
\end{bmatrix}
\]
\[
= |\psi(j\omega)|^2 w - |\psi(j\omega)|^2 X_{\tau_2} w + (\Delta_r(w))^* X_{\tau_2} w \ldots
\]
\[
+ w^* X_{\tau_2} (\Delta_r(w)) - (\Delta_r(w))^* X_{\tau_2} (\Delta_r(w))
\]
\[
= [s] \hat{\Pi}_{r_2} \begin{bmatrix} w \\
\Delta_r(w)
\end{bmatrix}
\]

In [11] a possible approximation $\psi(j\omega)$ that satisfies (5) is given by
\[
\psi(s) = \sqrt{\frac{8}{2-d}} \frac{\tau_0^2 s^2 + \gamma_0 s \sqrt{50}}{\tau_0^2 s^2 + \gamma_0 s \sqrt{6.5} + 2s \sqrt{50} + \sqrt{50}} + \delta.
\]

Note, that for the case of constant time delays, e.g. $d = 0$, (6) simplifies to the IQC given in [6]. The given functions $\psi(j\omega)$ in both cases are exactly the same, if we assume $d = 0$ and $\delta = 0$. Since $d$ appears only as a scaling in (6), $\psi(j\omega)$ for time-varying time delays is also only scaled, while poles and zeros do not differ from the constant case.

B. Robust Stability Analysis with IQCs

The goal here is to analyze the stability of feedback loops as shown in Fig. 1 using the conditions given by Lemma 1. In general, a parameterization of the IQC multiplier $\Pi$ is chosen in accordance with the operator, such that condition (ii) is trivially fulfilled. As described in Section II-A, in general a structure for the multiplier $\Pi$ is chosen such that it always fulfills the condition (ii) in Lemma 1, since solving the integral quadratic constraint in (1) is not trivial. In the case of frequency independent operators, static multipliers can be used, which lead to problems that can be efficiently solved as LMI. Thus no structure for $\Pi$ has to be enforced in advance.

Assume that $\Pi$ is a real-rational proper multiplier with no poles on the imaginary axis. Then as shown in [14] $\Pi$ can be factorized into a static symmetric part $M$ and a proper stable transfer function $\Psi$ as
\[
\Pi = \Psi^* M \Psi = [s]^* \begin{bmatrix}
M_{11} & M_{12} \\
M_{T_2}^T & M_{T_22}
\end{bmatrix} \begin{bmatrix}
\Psi_{11} & \Psi_{12} \\
0 & \Psi_{22}
\end{bmatrix}
\]

where $M$ and $\Psi$ are partitioned conformally with to $[G I]^T$. We define $\Psi_{11} = \begin{bmatrix} A_{11} & B_{11} \\
C_{11} & D_{11}
\end{bmatrix}$ and $\Psi_{12}$ and $\Psi_{22}$ accordingly. In the following the dynamic multiplier part is absorbed in $[G I]^T$, resulting in
\[
G = \begin{bmatrix}
A & B \\
0 & D
\end{bmatrix} = \Psi \begin{bmatrix} G \\
I
\end{bmatrix}
\]

Note that if $\Pi$ is structured as
\[
\Pi = \mathcal{P} \begin{bmatrix}
\operatorname{diag} K & \sum_{i=1}^{p_k} \Pi_k^i
\end{bmatrix}
\]

(8)
The interconnected system is described by condition (i) of Lemma 1 holds, if there exist matrices \( K \) and \( N \) by a state-space representation of the form:

\[
\dot{x} = A x + B_1 q + B_2 w, \quad y = C_1 x + D_{12} q + D_{1p} w, \quad z = C_2 x + D_{22} q + D_{2p} w
\]

Consider a general system of \( N \) interconnected LPV subsystems as shown in Fig. 2. Each subsystem is described by a state-space representation of the form:

\[
\dot{x}^k = A^k x^k + B_1^k q^k + B_2^k w^k, \quad y^k = C_1^k x^k + D_{12}^k q^k + D_{1p}^k w^k, \quad z^k = C_2^k x^k + D_{22}^k q^k + D_{2p}^k w^k
\]

The interconnection system is described by

\[
\dot{z} = \Theta h, \quad q = \Theta_1 h, \quad d = \Theta_2 h
\]

where \( k = \{1, \ldots, N\} \) and

\[
x = \text{col}_{k=1}^N (x^k) \in \mathbb{R}^{n_x}, \quad h = \text{col}_{k=1}^N (h^k) \in \mathbb{R}^{n_h},
q = \text{col}_{k=1}^N (q^k) \in \mathbb{R}^{n_q}, \quad w = \text{col}_{k=1}^N (w^k) \in \mathbb{R}^{n_w},

v = \text{col}_{k=1}^N (v^k) \in \mathbb{R}^{n_v}, \quad z = \text{col}_{k=1}^N (z^k) \in \mathbb{R}^{n_z},
\]

The interconnection is modeled by the operator \( P \), which can be partitioned as

\[
P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1N} \\ P_{21} & P_{22} & \cdots & P_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ P_{N1} & P_{N2} & \cdots & P_{NN} \end{bmatrix}, \quad \text{with} \quad P_{ij} : \mathbb{R}^{n_j} \to \mathbb{R}^{n_i}.
\]

We allow \( P \) to be a nonlinear dynamic or time-varying operator and thus note, that the above formulation of interconnected LPV systems, encompasses the form proposed in [3]. In fact, the representation proposed here is more compact. The compactness of the representation arises from the fact, that we do not require the interconnection operator \( P \) to be of diagonal structure. In cases, where a subsystem’s output signals are received by different subsystems, or linear combinations of several subsystem’s output signals are received via a single input channel of another subsystem, this would effectively increase the size of the system matrices, as output and input signals would have to occur repeatedly in \( w \) and \( v \), respectively. We further motivate the above representation by showing its connection to recent work on decomposable systems [4], a special case of interconnected systems, for which controller synthesis can be performed particularly efficiently.

A. Special Interconnection Structures

Several special cases are of interest, which considerably affect the complexity of the synthesis approaches for distributed LPV controllers. Let us consider a factorization, which associates an operator with each interconnection in- and output. We can then factorize the interconnection operators without loss of generality as \( P_{ij} = Y_i A_{ij} U_j \),

with \( A_{ij} : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}, \quad U_j : \mathbb{R}^{n_j} \to \mathbb{R}^{n_j}, \quad Y_j : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i} \).

We have \( P = YAU \),

with \( A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix}, \quad U = \bigoplus_{j=1}^N U_j, \quad Y = \bigoplus_{i=1}^N Y_i.
\)

We call \( A \) the adjacency operator, whereas we call \( Y, U \) the interconnection in- and output operators, respectively.

1) Square interconnection operators \( P_{ij} \): In the case of square interconnection operators \( P_{ij} : \mathbb{R}^{n_P} \to \mathbb{R}^{n_P} \), we require all subsystems to have the same number of interconnection in- and output channels \( n_P \). This holds true in many cases, e.g. when we are dealing with multi-agent systems of identical subsystems. Furthermore, empty zero channels can be added to meet this condition, which is a prerequisite for the following special cases.
a) Scalar repeated adjacency operators: Scalar repeated adjacency operators occur, when the same scalar interconnection adjacency operator \( a_{ij} \) is encoding information on the connection between two subsystems, irrespective of the individual input and output channels:

\[ A_{ij} = a_{ij} I_{n_p}, \quad \text{with} \quad a_{ij} : \mathbb{R} \to \mathbb{R}. \]

In such cases, we can write

\[ A = a \otimes I_{n_p}, \quad \text{with} \quad a = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1N} \\ a_{21} & a_{22} & \ldots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \ldots & a_{NN} \end{bmatrix}. \]

If the adjacency operators are real, then the interconnection topology can often be defined by an adjacency matrix or a Laplacian.

b) Scalar repeated interconnection operators: Analogously, one may assume, that all interconnection input- and output operators are identical among all the channels of a given subsystem:

\[ U_j = u_j I_{n_p}, \quad u = \bigoplus_{j=1}^{N} u_j, \quad Y_i = y_i I_{n_p}, \quad y = \bigoplus_{i=1}^{N} y_i. \]

Consequently, with both scalar repeated adjacency and interconnection operators we can write

\[ P = (y \otimes I_{n_p}) (a \otimes I_{n_p}) (u \otimes I_{n_p}) = (ya u) \otimes I_{n_p}. \]

B. Detailed LFT Representation of Interconnected Systems

The above factorization of the interconnection operator is motivated by the fact, that different types of nonlinear dynamic or time-varying operators require different integral quadratic constraints (IQCs) in order to perform stability analysis or synthesis. For this purpose, one can introduce separate LFT channels for each operator \( Y, A \) and \( U \). The interconnection input, adjacency and output operator channels are defined by \( v_y = Y(w_y), \quad v_a = A(w_a) \) and \( v_u = U(w_u) \), respectively. This leads to the representation

\[ T : \begin{bmatrix} \dot{x} \\ \dot{h} \\ w_y \\ w_a \\ w_u \\ q \end{bmatrix} = \begin{bmatrix} A & B_0 & 0 & 0 & 0 & 0 \\ C_\Theta & D_{\Theta \Theta} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} x \\ q \\ v_y \\ v_a \\ v_u \end{bmatrix}, \]

\[ q = \Theta h, \quad v_y = Y(w_y), \quad v_a = A(w_a), \quad v_u = U(w_u). \]

C. Interconnected Systems with Identical Subsystems

In many cases, e.g. when multi-agent systems are considered, the subsystems are assumed to have identical LTI dynamics, whereas the LPV scheduling blocks are identically structured, but can take different values. Thus, the individual agents \( \mathcal{H}_i, \quad k = 1, \ldots, N \) share the same state-space matrices, such that, e.g. \( A^1 = \cdots = A^N := A \). This in turn allows to write all matrices of the interconnected system as e.g. \( A = I_N \otimes A \).

1) Connection to Decomposable Systems: For decomposable systems \([4, 5, 15, 16, 17]\) identical subsystems are assumed. Thus the previous simplifications hold. It is further assumed that the interconnection is square and real and represents the communication structure via an adjacency matrix or Laplacian, as well as ideal interconnection in- and output operators, given by \( y = I_N \) and \( u = I_N \). Note that if one of the interconnection operator is \( I \) its is not necessary to extract an extra LFT channel for it.

IV. ROBUST STABILITY ANALYSIS FOR INTERCONNECTED SYSTEMS

Given is the transfer function representation of (14) as

\[ G = \begin{bmatrix} G_{\Theta \Theta} & G_{\Theta y} & G_{\Theta a} \\ G_{\Theta y \Theta} & G_{\Theta y a} & G_{\Theta y u} \\ G_{\Theta a \Theta} & G_{\Theta a u} & G_{\Theta a u} \end{bmatrix}, \quad \begin{bmatrix} h \\ w_y \\ w_a \end{bmatrix} = \begin{bmatrix} G \\ v_y \\ v_a \end{bmatrix}. \]

Each channel \( \Theta, y, a, u \) describes a different type of uncertainty and thus the corresponding signals have to satisfy IQCs defined by different multipliers \( \Pi^\Theta, \Pi^y, \Pi^a \) and \( \Pi^u \).

The \( z \)-channel here is the seen as the performance channel. There exist different IQCs for different performance specifications. Performance in the sense of the \( L_2 \) gain is further exploited in the following. Here the \( z \)-channel will be treated in exactly the same way as the LFT-channels, such that the corresponding signals have to satisfy IQCs with the multiplier \( \Pi^z \).

Remark: Consider signals \( w_k \in \mathcal{L}^2_s[0, \infty) \) and \( v^k = \Delta^k(w^k) \in \mathcal{L}^2_s[0, \infty) \) for \( k = 1, \ldots, K \), where each signal pair satisfies the IQC as in Definition 1 with \( \Pi^k \).

Then the operator \( \Delta = \text{diag}_{k=1}^{K} (\Delta^k) \) with the signals \( w = \text{col}_{k=1}^{K}(w^k) \) and \( v = \text{col}_{k=1}^{K}(v^k) = \Delta(w) \) satisfy the IQC in Definition 1 with

\[ \Pi = \mathcal{P}^{T} \left[ \text{diag}_{k=1}^{K} (\Pi^k) \right] \mathcal{P} \]

with \( \mathcal{P} = \begin{bmatrix} I_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & I_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & I_{m-2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & I_{m-2} & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I_{m-1} \end{bmatrix} \)

As pointed out in [11] if \( \Pi^k \) for \( i = 1, \ldots, p_k \) satisfies (1) for the signals \( w^k \) and \( v^k \), its affine combination \( \Pi^k = x^1 \Pi^k_1 + \cdots + x^p_k \Pi^k_{p_k} \) also does.

Proposition 3: With the IQCs given in the previous paragraph for the single channels, according to the previous remark the IQC for the whole system (15) is satisfied by

\[ \Pi = \mathcal{P}_G^{T} \left[ \text{diag} (\Pi^\Theta, \Pi^y, \Pi^a, \Pi^u, \Pi^z) \right] \mathcal{P}_G = \mathcal{P}_G^{T} \Pi \mathcal{P}_G \]
A. Robust Stability Analysis for Interconnected Systems with Time Delays

In the following it is assumed that the system is interconnected by a square interconnection operator with scalar repeated adjacency operator and interconnection operators. Here the interconnection operator $A$ is static and described by the Laplacian and the input and output operators $Y$ and $U$ are dynamic and represent time delays as

$$Y = \left[ \begin{array}{c} \text{diag} \\ \tau_k \end{array} \left( \Delta_\gamma \right) \right] \otimes I_{n,v}$$

with $0 \leq \tau_k^2(t) \leq \tau_0 \hspace{1em} |\tau_k^2(t)| \leq d \hspace{1em} \forall k = 1, \ldots, N$ (17) and $U$ defined respectively.

The LPV channel is assumed to contain a real time-varying uncertainty.

1) IQCs for $Y$ and $U$: Due to the block-diagonal structure of $Y$ as given in (17), $Y = Y(w)$ can be partitioned in $v_i = \Delta_\gamma(w_i)$ for all $k = 1, \ldots, N$. According to Section II-A,2 such signals satisfy the IQC in (1) with $\Pi^\delta_k = \Pi^\delta_{\tau_1} + \Pi^\delta_{\tau_2}$ with the factorizations $\Pi^\delta_{\tau_1} = M^\delta_{\tau_1} \Psi^\delta_{\tau_1} M^\delta_{\tau_1}$ and $\Pi^\delta_{\tau_2}$ respectively. With (16) the IQC for the uncertain operator $Y$ is given as in (8) by

$$\Pi^\delta_{\tau_1} = \mathcal{P}^T \left( \Pi^\delta_{\tau_1} \Pi^\delta_{\tau_2} \right) \mathcal{P},$$

factorized according to (9). A more compact factorization is however possible if both summands are factorized together leading to

$$\Pi^\delta_{\tau_1} + \Pi^\delta_{\tau_2} = [s]^T \left[ \begin{array}{cc} X^\delta_{\tau_1} & 0 \\ 0 & -X^\delta_{\tau_2} + \frac{1}{2d} X^\gamma_{\tau_1} + X^\gamma_{\tau_2} \\ 0 \end{array} \right] \left[ \begin{array}{c} \psi \mid 1 \end{array} \right].$$

For $U$ the same considerations hold true leading to $\Pi^u$.

2) IQCs for $\Theta$: Due to the diagonal structure of $\Theta$ the multiplier $\Pi^\Theta_k$ can be constructed as in (8). In the case of constant uncertainties, constant multipliers such as $[s]^T \left[ \begin{array}{cc} R^\Theta_k & S^\Theta_k \\ \ast & Q^\Theta_k \end{array} \right]$ are used and (1) simplifies to

$$[s]^T \Pi^\Theta_k \left[ \begin{array}{c} I \\ \Theta_k \end{array} \right] > 0 \hspace{1em} \forall k = 1, \ldots, N.$$ For the static case the condition leads to the given LMI condition and thus can easily be checked. There are however approaches on the structure of the multiplier to avoid that extra condition but conservatism is introduced [18].

3) IQCs for $A$: Here the uncertainty $A$ is assumed to be static. Analogously to the LPV uncertainty, a static multiplier can be used, such as $\Pi^a = \left[ \begin{array}{cc} R^a \otimes S^a \\ \ast \otimes Q^a \end{array} \right]$ with the additional condition $[s]^T \Pi^a \left[ \begin{array}{c} I \\ \Theta_k \end{array} \right] > 0$. Note that due to the diagonal structure of $\Theta$, and thus of $\Pi^\Theta$, in contrast to $\Pi^a$, the number of unknowns of the multiplier for $\Theta$ scales linearly with $N$, that of $A$ scales quadratically.

Particularly efficient approaches are available for individual cases, e.g. when $a$ is static or time-varying and/or symmetric. In depth discussions can be found in [5], [15], [16].

4) IQCs for the $z$-channel: The performance of the system $G$ in terms of the $L_2$ gain between $d$ and $z$ is less than $\gamma > 0$ if the signals satisfy the IQC with $\Pi^z = \text{diag}(I, -\gamma^2 I)$, [19].

B. Simplifying assumptions on IQC multiplier

In the following structural constraints on the multipliers are proposed, that eliminate the dependence of the analysis on the number of subsystems $N$. The conditions are particularly efficient to solve, when interconnected systems with identical subsystems are considered, although some degree of conservatism is introduced.

Bounds on the time delay and its rate of change are assumed to be identical for each time delay operator, such that the corresponding multiplier can be chosen as $\Pi^y = I_N \otimes \Pi^y$. The same holds for the LPV uncertainty channel, as their admissible parameter range is assumed to be identical among the subsystems, leading to $\Pi^\delta = I_N \otimes \Pi^\delta$. Regarding the additional condition we therefore choose to impose LMI constraints on the vertices of the convex hull of the admissible LPV parameter range.

For the operator $A$, which itself is not block-diagonal, we propose to choose the multiplier constrained to a block-diagonal structure [3], [5], [16] as $\Pi^a = \left[ I_N \otimes R^a I_N \otimes S^a \right] \ast I_N \otimes Q^a$ while the additional condition does not change.

For interconnected systems with equal subsystems, as discussed in Section III-C all system matrices in (14) are block-diagonal with $N$ equal blocks. With the proposed simplifying assumptions all multipliers have the same structure as well, such that the inequalities (10), (11) and (12) can be solved for one single subsystem. That reduces complexity enormously, especially for large scale interconnected systems. Note that without additional assumptions the additional multiplier for $A$ is still scaled with $N$. There are however different approaches for different cases of $A$. One case is discussed in the numerical example in Section V.

V. Numerical Example

In the following a numerical example is considered, that is partly adopted from [11] with an additional interconnection structure. We consider an interconnected LTI system with $N$ identical subsystems, thus there is no LPV uncertainty. The LFT-channel for the output operator is also not needed here since $U = I$. The input operator $Y$ is assumed to be a scalar repeated time-varying time delay, where the delay and its rate of change for the different subsystems may differ, but are upper bounded by the same $\tau_0$ and $d$, respectively.

Here, the adjacency operator is given by a time-varying...
As shown in [5], due to the time-varying and symmetric nature of the adjacency operator, we can apply the similarity transformation \( \Lambda \otimes I_{n_y} \) to the additional constraint for the adjacency operator with \( \Lambda^{-1} a \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \). Due to the symmetry the applied transformation is equal to a congruence transformation and thus does not change the result of the LMI. This transformation block-diagonalizes the additional condition for the adjacency operator, that can then be decomposed as

\[
[x]^T \begin{bmatrix} Q^a \, S^a \\ * \, Q^a \end{bmatrix} I_L > 0 \quad \text{for } \lambda = \{0, \lambda_{\max}\}
\]

with \( \lambda_{\max} = \max \text{eig} (L(t), \forall t) \). Note that the minimal eigenvalue of \( L(t) \) is 0.

The system with the additional LFT-channels is given by

\[
\begin{bmatrix} \dot{x}^k \\ u^k \\ w^k \\ v^k \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -0.9 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^k \\ u^k \\ w^k \\ v^k \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^k,
\]

\[
Y = \frac{\partial}{\partial x} A(w_y), \quad v_a = A(w_a).
\]

with \( Y \) as defined in (17) and \( A \) as defined above.

In Fig. 3, configurations for which stability has been proven by the analysis condition in Lemma 2 are shown. The region to the lower left of the curves corresponding to a fixed value of \( d \) indicate parameters \( \lambda_{\max} \) and \( \gamma_0 \) for which stability guarantees have been established. The parameter \( \lambda_{\max} \) is a measure for the connectedness of the subsystems. It is evident that the more the system is connected, the smaller is the allowed delay, since it is the interconnection channel, which is delayed. If \( \lambda_{\max} \) approaches 0, \( \gamma_0 \) goes to infinity (not visible in the figure), since the system without any interconnection is stable. The slower the allowed rate of change in \( \tau \), the larger values of \( \lambda_{\max} \) and \( \gamma_0 \) for which stability guarantees can be established.

Fig. 4 shows the performance bound \( \gamma \). If the values of \( \lambda_{\max} \) and \( \gamma_0 \) approach a configuration near to the curve that indicates the boundary for which a stability certificate can be found, \( \gamma \) grows very large.

VI. CONCLUSION

In this work we propose a compact framework to model interconnected LPV systems, for which the notion of decomposable LPV systems can be identified as a special case. A convex analysis condition for interconnections with uncertain time-varying communication delays and switching communication topologies based on IQCs is proposed and illustrated with a numerical example. The presented framework is not restricted to uncertain time-varying time delays, but can easily be adopted to other uncertainties.

REFERENCES


